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Complete Characterization of all Discrete-Time Linear Systems of Convolution Type

Holger Boche and Gunther Reißig
Technische Universität Dresden
Institut für Grundlagen der Elektrotechnik und Elektronik

Mommsenstr. 13, D-01062 Dresden, Germany
E-mail: boche@iee1.et.tu-dresden.de

Abstract

In this paper the behaviour of discrete-time linear systems is investigated. A complete characterization of all discrete-time linear systems of convolution type is given in the paper. The result is a generalization of results given by S. Boyd and I. Sandberg.

1 Introduction

In this paper the behaviour of discrete-time linear systems is investigated. l^∞ is the space of bounded discrete-time functions, endowed with the norm $\|x\| = \sup_{k\in\mathbb{Z}} |x(k)|$. Let denote l_0^∞ the space of all bounded discrete-time functions $x \in l^\infty$ that satisfy $\lim_{|k|\to\infty} x(k) = 0$.
All investigated systems are BIBO- stable (bounded input-bounded output). This means that a positive number C_1 exists, so that

$$\|Sx\| \leq C_1 \tag{1}$$

holds for all $x \in l^\infty$, $\|x\| \leq 1$. The main idea of the theory of discrete-time single-input single-output linear systems S is that every such system has an input-output map that can be represented by an expression of the form

$$y(k) = (Sx)(k) = \sum_{l=-\infty}^{\infty} h(k,l)x(l)\ , \tag{2}$$

where x is the input, h the inpulse response, and y the output of the system S.
Systems S of this type are called **systems of sum type**.
If the system S is time-invariant, then equation (2) reduces to

$$y(k) = (Sx)(k) = \sum_{l=-\infty}^{\infty} h(k-l)x(l)\ . \tag{3}$$

Systems of this kind are called **systems of convolution type**. They are especially important for signal processing [3].
The function h in (3) is the impulse response of the system S, i.e., $h(l) = (S\delta)(l)$, where δ denotes the Kroneker-Delta-function

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0\ . \end{cases}$$

Often it is asserted that the representations (2) and (3) hold for all linear discrete-time systems [3]. It was recently discovered that not all discrete-time single-input single-output linear systems are of sum type or convolution type [2], [4], [5]. Some examples were constructed in [2], [4].
A complete characterization of BIBO-stable discrete-time systems was given in [1]. Let $E_n x$ denote the signal given by

$$(E_n x)(k) = \begin{cases} x(k) & |k| > n \\ 0 & |k| \leq n\ . \end{cases}$$

The following was shown in [1].

Theorem 1 *Let S be a BIBO-stable linear system. Then the representation*

$$(Sx)(k) = (S_1 x)(k) + (S_\infty x)(k) \qquad (4)$$

holds for all signals $x \in l^\infty$, where the system S_1 has the form

$$(S_1 x)(k) = \sum_{l=-\infty}^{\infty} h(k,l)x(l) \ , \qquad (5)$$

and the system S_∞ is defined by

$$(S_\infty x)(k) = \lim_{n\to\infty} \left((SE_n x)(k) \right) \ . \qquad (6)$$

Both systems S_1 and S_∞ are linear and BIBO-stable. If the system S is a LTI-system then the systems S_1 and S_∞ are also LTI-systems.

The limit in (6) exists for all functions $x \in l^\infty$. Representation (4) means that each BIBO-stable linear system can be split into two subsystems S_1 and S_∞ which are given by (5) and (6). System S_1 is the only system of sum type, that means the system S is of sum type iff $S_\infty = 0$. The impulse response of the system S_∞ is zero for all $k \in \mathbb{Z}$. For all signals $x \in l_0^\infty$ we have

$$(Sx)(k) = (S_1 x)(k)$$

for all $k \in \mathbb{Z}$. The BIBO-Norm $\|S\|_B$ of linear system S is the infimum of all numbers C_1, so that (1) holds. The representation

$$\|S\|_B = \|S_1\|_B + \|S_\infty\|_B \qquad (7)$$

for the BIBO-norm $\|S\|_B$ of an LTI-system S was shown in [1].

2 Main result

Since not all discrete-time single-input single-output linear systems S are of sum type or convolution type it would be interesting to characterize these systems.
This means to find all discrete-time linear systems S, so that the identity $S = S_1$, or equivalently, $S_\infty = 0$ holds. That means $S_\infty x = 0$ for all signals $x \in l^\infty$.
For this reason it is necessary to introduce one more concept. A sequence of bounded input-signals x_n, $n \in \mathbb{N}$, is said to be **converge** to a bounded signal x, if a positive number C_2 exists, so that

$$|x_n(k)| \le C_2 \qquad (8)$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} x_n(k) = x(k) \qquad (9)$$

for all $k \in \mathbb{Z}$ holds. A linear discrete-time BIBO-stable system S is said to be **strongly continuous**, if for each convergent sequence the equation

$$\lim_{n\to\infty} (Sx_n)(k) = (Sx)(k) \ , k \in \mathbb{Z} \ , \qquad (10)$$

holds. This means the sequence of output-signals $y_n = Sx_n$ converges to the output-signal $y = Sx$.
Some clarifications are necessary. There is another important convergence concept. A sequence of bounded input-signals x_n, $n \in \mathbb{N}$, **converges in norm** to a bounded signal x, if

$$\lim_{n\to\infty} \|x - x_n\| = 0 \qquad (11)$$

holds. Every norm-convergent sequence of bounded signals is convergent but not vice-versa. This means the concept convergence is weaker than the concept norm-convergence. There is also a second definition of continuous linear systems. A linear system S is said to be **continuous**, if for each sequence x_n, $n \in \mathbb{N}$, with (11) the equation

$$\lim_{n\to\infty} \|Sx - Sx_n\| = 0 \qquad (12)$$

holds. Each BIBO-stable linear system S is continuous. However, the concept strongly continuous is stronger than the concept continuous.
The following is our main result.

Theorem 2 *A linear BIBO-stable discrete-time system S is a system of sum type, if and only if the system S is strongly continuous.*
A BIBO-stable LTI-system S is a system of convolution type, if and only if the system S is strongly continuous.

3 Proof of the main Result

In this section we give a proof of our main result.
Proof: (Theorem 2) Suppose that the BIBO-stable LTI-system S is strongly continuous. For each $n \in I\!N$ we construct a signal x_n by

$$x_n(k) = \begin{cases} x(k) & |k| \leq n \\ 0 & |k| > n \ . \end{cases} \tag{13}$$

For these signals x_n we have $\|x_n\| \leq \|x\|$ and

$$x(k) = \lim_{n\to\infty} x_n(k) \tag{14}$$

for all $k \in Z\!\!\!Z$. That means the sequence of signals x_n converges to the signal x. Since the system S is strongly continuous we have for all $k \in Z\!\!\!Z$

$$(Sx)(k) = (Sx_n)(k) \ . \tag{15}$$

For the signals x_n the representation

$$x_n(k) = \sum_{l=-n}^{n} x(l)\delta(k-l) \tag{16}$$

holds. This gives

$$(Sx_n)(k) = \sum_{l=-n}^{n} x(l)h(k-l) \tag{17}$$

and

$$\begin{aligned} (Sx)(k) &= \lim_{n\to\infty} \sum_{l=-n}^{n} x(l)h(k-l) \\ &= \sum_{l=-\infty}^{\infty} x(l)h(k-l) \\ &= (S_1 x)(k) \ . \end{aligned} \tag{18}$$

Since $x \in l^\infty$ was an abitriary signal, we have $S = S_1$, and the system S is a system of convolution type.
Let now S be a system of convolution type and x_n be a convergent sequence of signals. We have to show, that the system S is strongly continuous.
Let $\varepsilon > 0$ be an abitriary number. Since the system S is BIBO-stable there exists a number M such that

$$\sum_{|l|\geq M} |h(l)| < \varepsilon \ . \tag{19}$$

Since the sequence of signals x_n is a convergence sequence there exists a positive number C_3 such that

$$|x_n(k)| \leq C_3 \tag{20}$$

holds. With the signal x_n we have

$$\begin{aligned} &|(Sx)(k) - (Sx_n)(k)| = \\ &= \left| \sum_{l=-\infty}^{\infty} h(l)(x(k-l) - x_n(k-l)) \right| \\ &\leq \sum_{l=-\infty}^{\infty} |h(l)| \cdot |x(k-l) - x_n(k-l)| \\ &= \sum_{l=-M}^{M} |h(l)| \cdot |x(k-l) - x_n(k-l)| + \\ &\quad + \sum_{|l|>M} |h(l)| \cdot |x(k-l) - x_n(k-l)| \ . \end{aligned} \tag{21}$$

At next we investigate the second term on the right side of (21). We have

$$\begin{aligned} &\sum_{|l|>M} |h(l)| \cdot |x(k-l) - x_n(k-l)| \leq \\ &\leq \sum_{|l|>M} |h(l)| \cdot (|x(k-l)| + |x_n(k-l)|) \\ &\leq 2C_3 \cdot \sum_{|l|>M} |h(l)| < 2C_3 \cdot \varepsilon \ . \end{aligned} \tag{22}$$

For the first term on the right side of (21) we have

$$\sum_{l=-M}^{M} |h(l)| \cdot |x(k-l) - x_n(k-l)| \leq$$

$$
\begin{aligned}
&\leq \max_{|l|\leq M} |x(k-l)-x_n(k-l)| \cdot \sum_{l=-M}^{M} |h(l)| \\
&\leq \max_{|l|\leq M} |x(k-l)-x_n(k-l)| \cdot \sum_{l=-\infty}^{\infty} |h(l)| \\
&= C_4 \cdot \max_{|l|\leq M} |x(k-l)-x_n(k-l)| \; . \qquad (23)
\end{aligned}
$$

For each number l, $|l| \leq M$, there exists a number $n_0 = n_0(\varepsilon, l)$ such that for all numbers $n \geq n_0$ the inequality

$$|x(k-l) - x_n(k-l)| < \varepsilon \qquad (24)$$

holds. Now define the number $n_* = \max_{|l|\leq M} n_0(\varepsilon, l)$. For all numbers $n \geq n_*$ we have

$$\max_{|l|\leq M} |x(k-l) - x_n(k-l)| < \varepsilon \; . \qquad (25)$$

This yields for all numbers $n \geq n_*$

$$|(Sx)(k) - (Sx_n)(k)| \leq \varepsilon \cdot \Big(2C_3 + C_4\Big) \; . \qquad (26)$$

Since the inequality (26) holds for all $\varepsilon > 0$ we have

$$(Sx)(k) = \lim_{n\to\infty} (Sx_n)(k) \; . \qquad (27)$$

So the system S is strongly continuous.
This proofs Theorem 2 for BIBO-stable LTI-systems. The proof of the general result is similar.

4 BIBO-norm

In this section a characterization with respect to the BIBO-norm of BIBO-stable LTI-systems of convolution type is given. Most of the results are only valid for LTI-systems. For abitriary BIBO-stable linear systems the behaviour of the BIBO-norm of these systems is more diffucult. Let l_c^∞ denote the set of all signals x with are only non zero for finite time intervals. The next theorem gives a characterization of the behaviour of the BIBO-norm $\|S_\infty\|_B$. The system S is an abitriary BIBO-stable linear system.

Theorem 3 *The BIBO-stable LTI-system S is a system of convolution type if and only if the representation*

$$\|S\|_B = \sup_{\|x\|\leq 1; x\in l_c^\infty} \|Sx\| \qquad (28)$$

holds.

Proof: Let S be a system of convolution type. So we have $S = S_1$. Consider for each $K \in \mathbb{N}$ the signal

$$x_K(l) = \begin{cases} sign(h(-l)) & |l| \leq K \\ 0 & |l| > K \; . \end{cases} \qquad (29)$$

We have $x_K \in l_c^\infty$. With this signal we have

$$
\begin{aligned}
\sum_{l=-K}^{K} x(l)h(-l) &= \sum_{l=-K}^{K} |h(l)| = (Sx_K)(0) \\
&\leq \|Sx_K\| \leq \|S\|_B \cdot \|x_K\| \\
&= \|S\|_B \leq \sum_{l=-\infty}^{\infty} |h(l)| \; . \qquad (30)
\end{aligned}
$$

This gives for all $K \in \mathbb{Z}$

$$
\begin{aligned}
\sum_{l=-K}^{K} |h(l)| &\leq \sup_{\|x\|\leq 1; x\in l_c^\infty} \|Sx\| \\
&\leq \|S\|_B \leq \sum_{l=-\infty}^{\infty} |h(l)| \; . \qquad (31)
\end{aligned}
$$

Since the inequality (31) holds for all K we have (28). Now suppose that the equation (28) holds. We have to show that the system S is a system of convolution type. We have already shown that the equation

$$\|S_1\|_B = \sup_{\|x\|\leq 1; x\in l_c^\infty} \|S_1 x\| \qquad (32)$$

holds. Let $x \in l_c^\infty$ be an abitriary signal. There exists a number $N \in \mathbb{N}$ such that

$$x(k) = 0$$

for all $|k| > N$. So we have

$$(Sx)(k) = \sum_{k=-N}^{N} x(l)h(k-l) = (S_1 x)(k) \ . \quad (33)$$

With (28) this gives

$$\|S\|_B = \|S_1\|_B \quad (34)$$

and so

$$\|S_\infty\|_B = 0 \quad (35)$$

holds.

With the help of Theorem 3 we have the result, that a BIBO-stable LTI-system S is a system of convolution type if and only if the BIBO-norm of the system S can be calculated by

$$\|S\|_B = \sup_{\|x\|\le 1; x\in l_0^\infty} \|Sx\| \ . \quad (36)$$

In the next theorem we consider abitriary BIBO-stable linear systems.

Theorem 4 *For the BIBO-norm $\|S\|_B$ of a BIBO-stable linear system S the representation (7) holds if and only if for each $\varepsilon > 0$ there exist a signal $x_\varepsilon \in l^\infty$ with $\|x_\varepsilon\| = 1$ and a number $k_\varepsilon \in \mathbb{Z}$ such that*

$$(S_1 x_\varepsilon)(k_\varepsilon) > \|S_1\|_B - \varepsilon \quad (37)$$

and

$$(S_\infty x_\varepsilon)(k_\varepsilon) > \|S_\infty\|_B - \varepsilon \quad (38)$$

holds.

Proof: Suppose that (37) and (38) holds. Then we have

$$\begin{aligned}(Sx_\varepsilon)(k_\varepsilon) &= (S_1 x_\varepsilon)(k_\varepsilon) + (S_\infty x_\varepsilon)(k_\varepsilon) > \\ &> \|S_1\|_B + \|S_\infty\|_B - 2\varepsilon \ . \quad (39)\end{aligned}$$

This gives

$$\|S\|_B > \|S_1\|_B + \|S_\infty\|_B - 2\varepsilon \ . \quad (40)$$

Since the inequality (40) holds for all $\varepsilon > 0$ we have $\|S\|_B \ge \|S_1\|_B + \|S_\infty\|_B$. The inverse inequality $\|S\|_B \le \|S_1\|_B + \|S_\infty\|_B$ is also satisfied which gives (7).

Now suppose that (7) holds. There exist a signal $x_\varepsilon \in l^\infty$ with $\|x_\varepsilon\| = 1$ and a number $k_\varepsilon \in \mathbb{Z}$ such that

$$(Sx_\varepsilon)(k_\varepsilon) > \|S\|_B - \frac{\varepsilon}{2} \quad (41)$$

holds. We also have

$$(S_1 x_\varepsilon)(k_\varepsilon) + (S_\infty x_\varepsilon)(k_\varepsilon) \le (S_1 x_\varepsilon)(k_\varepsilon) + \|S_\infty\|_B \ . \quad (42)$$

This gives

$$(S_1 x_\varepsilon)(k_\varepsilon) \ge \|S\|_B - \frac{\varepsilon}{2} - \|S_\infty\|_B = \|S_1\|_B - \frac{\varepsilon}{2} \ . \quad (43)$$

By the same way we can show that

$$(S_\infty x_\varepsilon)(k_\varepsilon) \ge \|S_\infty\|_B - \frac{\varepsilon}{2} \quad (44)$$

holds. This proofs Theorem 4.

Theorem 5 *Let Q_n be the set of all signals $x \in l^\infty$ such that $x(k) = 0$ for all $|k| \le n$. We have for all $n \in \mathbb{N}$*

$$\|S_\infty\|_B = \sup_{\|x\|\le 1; x\in Q_n} \|S_\infty x\| \ . \quad (45)$$

Proof: Since $Q_n \subset l^\infty$ holds, we have

$$\sup_{\|x\|\le 1; x\in Q_n} \|S_\infty x\| \le \|S_\infty\|_B \ . \quad (46)$$

Let n be an abitriary number. We consider the signal x_ε such that (38) holds. We have

$$(S_\infty x_\varepsilon)(k_\varepsilon) = \lim_{m\to\infty} (SE_m x_\varepsilon)(k_\varepsilon) \ . \quad (47)$$

Let $m_0 \ge n$ be abitriary and $x_{\varepsilon,m_0} = E_{m_0} x_\varepsilon$. Then we have $x_{\varepsilon,m_0} \in Q_n$ and

$$(S_\infty x_\varepsilon)(k_\varepsilon) = (S_\infty x_{\varepsilon,m_0})(k_\varepsilon) \ . \quad (48)$$

This gives

$$\begin{aligned}\sup_{\|x\|\le 1; x\in Q_n} \|S_\infty x\| &\ge \|S_\infty x_{\varepsilon,m_0}\| \ge \\ \ge \ & (S_\infty x_{\varepsilon,m_0})(k_\varepsilon) \ge \|S_\infty\|_B - \varepsilon \ . \quad (49)\end{aligned}$$

Since the inequality (49) holds for all $\varepsilon > 0$ we have

$$\sup_{\|x\|\leq 1; x\in Q_n} \|S_\infty x\| \geq \|S_\infty\|_B \tag{50}$$

which proofs Theroem 5.
With the help of the next theorem it is possible to calculate the BIBO-norm $\|S_\infty\|_B$.

Theorem 6 *Let S be a BIBO-stable LTI-system and let*

$$\|S\|_{B,n} = \sup_{\|x\|\leq 1; x\in Q_n} \|Sx\| \ . \tag{51}$$

Then we have

$$\|S_\infty\|_B = \lim_{n\to\infty} \|S\|_{B,n} \ . \tag{52}$$

Proof: For all $n \in I\!N$ we have $Q_{n+1} \subset Q_n$. This gives

$$\|S\|_{B,n} \geq \|S\|_{B,n+1} \ . \tag{53}$$

With (53) the limit

$$\|S\|_{B,*} = \lim_{n\to\infty} \|S\|_{B,n} \tag{54}$$

exists. We have to show that $\|S\|_{B,*} = \|S_\infty\|_B$ holds. We have for all signals $x \in Q_n$ with $\|x\| \leq 1$

$$\begin{aligned} \|S_\infty x\| &\geq \|Sx\| - \|S_1 x\| \\ &\geq \|Sx\| - \sum_{|l|>n} |h(l)| \ . \end{aligned} \tag{55}$$

This gives

$$\|S_\infty\|_B \geq \|Sx\| - \sum_{|l|>n} |h(l)| \ . \tag{56}$$

The left side of the inequality (56) is independent with respect to the signal x. This gives

$$\|S_\infty\|_B \geq \|S\|_{B,n} - \sum_{|l|>n} |h(l)| \tag{57}$$

and so

$$\|S_\infty\|_B \geq \lim_{n\to\infty} \left(\|S\|_{B,n} - \sum_{|l|>n} |h(l)| \right) = \|S\|_{B,*} \ .$$

We also have for all $x \in Q_n$ with $\|x\| \leq 1$

$$\begin{aligned} \|S_\infty x\| &\leq \|Sx\| + \|S_1 x\| \\ &\geq \|Sx\| + \sum_{|l|>n} |h(l)| \ . \end{aligned} \tag{58}$$

This gives

$$\|S_\infty\|_B \geq \lim_{n\to\infty} \left(\|S\|_{B,n} + \sum_{|l|>n} |h(l)| \right) = \|S\|_{B,*} \tag{59}$$

which proofs the Theorem 6.
The follwing theorem is a consequence of the Theorem 6.

Theorem 7 *A BIBO-stable LTI-system is a system of convolution type if and only if the equation*

$$\lim_{n\to\infty} \|S\|_{B,n} = 0 \tag{60}$$

holds.

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