Behavior of Multitone Signals with Schroeder's Phase

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Abstract

The behavior of multitone signals with Schroeder's phase is investigated in this paper. Relations to the noise enhancement factor are analyzed. The results solve a problem proposed by Professor J. Massey.

Keywords: Multitone Signal, Crest Factor, Schroeder's Phase

1 Introduction

Multitone signal F is defined by

$$F(e^{j\omega}) = \frac{1}{\sqrt{N+1}} \cdot \sum_{k=0}^{N} f(k) e^{-jk\omega} \qquad (1)$$

with $|f(k)| = 1$ for $0 \leq k \leq N$ [2], [5], [6]. This means that there exists a real valued function ϕ such that $f(k) = e^{j\phi(k)}$ holds. The class of all multitone signals of the form (1) is denoted by $\mathcal{B}_N$. The crest factor of a continuous signal G is defined by

$$Cr(G) = \frac{\|G\|_\infty}{\|G\|_2} \qquad (2)$$

where $\|G\|_\infty$ is the peak value of the signal G, i.e.,

$$\|G\|_\infty = \max_{\omega \in [-\pi,\pi)} |G(e^{j\omega})|$$

and

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 \, d\omega \right)^{\frac{1}{2}}$$

is the energy of the signal G. For the energy of a multitone signal F we have with Parseval's equation

$$\|F\|_2 = \left(\frac{1}{N+1} \cdot \sum_{k=0}^{N} |f(k)|^2 \right)^{\frac{1}{2}} = 1 \, .$$

This gives for the crest factor of a multitone signal F the equation $Cr(F) = \|F\|_\infty$.
It is very difficult to find multitone signals with low crest factors [2], [5]. Of course we have $Cr(F) \geq 1$ for all multitone signals.

2 Schroeder's Phase

In [3] [4] one is interested in length-N aperiodic bipolar sequences s where $s(l) \in \{\pm 1\}$ for $0 \leq l \leq N-1$ and $s(l) = 0$ otherwise. The aperiodic impulse response of the inverse filter h_s is defined by

$$\sum_{l=0}^{N-1} s(l) \cdot h_s(k-l) = \delta(k) \, . \qquad (3)$$

The Kronecker-Delta function is given by $\delta(k) = 1$ for $k = 0$ and $\delta(k) = 0$ for $k \neq 0$. The $\mathcal{Z}$-transform of the function s is defined by

$$S(e^{j\omega}) = \sum_{l=0}^{N-1} s(l) \cdot e^{-jl\omega} \, . \qquad (4)$$

We suppose that $|S(e^{j\omega})| > 0$ holds for all $\omega \in [-\pi, \pi)$. The frequency response H_s of the inverse filter is also a continuous function. We have

$$H_s(e^{j\omega}) = \sum_{l=-\infty}^{\infty} h_s(l) \cdot e^{-jl\omega} = \frac{1}{S(e^{j\omega})} \cdot \qquad (5)$$

Since H_s is a continuous function it has also a finite energy, i.e., the noise enhancement factor κ_s given by [3], [4]

$$\begin{aligned} \kappa_s &= N \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_s(e^{j\omega})|^2 \, d\omega \\ &= N \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|S(e^{j\omega})|^2} \, d\omega \qquad (6) \end{aligned}$$

is finite. In [3] [4] one is interested in bipolar sequences, which have noise enhancement factors as small as possible. The noise enhancement factor satisfies $\kappa_s \geq 1$. It follows that $\kappa_s = 1$ if and only if the sequence s has a perfect flat spectrum $|S(e^{j\omega})|^2 = N$. For $N \geq 2$ it is easy to show that κ_s is greater then 1. Thus a first practical criterion of goodness of the bipolar sequence of s is the smallness of [4]

$$\Delta_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|S(e^{j\omega})|^2 - 1 \right)^2 d\omega \; . \tag{7}$$

The investigation of the behavior of the quanteties Δ_N is a very difficult task [3]. The exact asymptotic behavior of the numbers Δ_s and κ_s are unknown [3]. It is the aim of this paper to investigate the behavior of the numbers (7) for multitone signals with Schroeder's phase. Let $N \in I\!N$ be an arbitrary number. We consider the phase function ϕ_{SCH} defined by

$$\phi_{SCH}(k) = \begin{cases} \frac{k(k-1)\pi}{N+1} & 0 \le k \le N \\ 0 & k \notin [0, N] \; . \end{cases} \tag{8}$$

This function was first introduced by M.R. Schroeder [6]. See also [5]. The behavior of the related multitone signal

$$P_N(e^{j\omega}) = \frac{1}{\sqrt{N+1}} \cdot \sum_{k=0}^{N} \exp\left(j\phi_{SCH}(k) \right) \cdot e^{-jk\omega} \tag{9}$$

is now analyced. We consider the number

$$\Delta_N = \inf_{P \in \mathcal{B}_N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|P(e^{j\omega})|^2 - 1 \right)^2 d\omega \; . \tag{10}$$

The exact asymptotic behaviour of the number is unknown

It is shown in the next section that

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|P_N(e^{j\omega})|^2 - 1 \right)^2 d\omega = 0 \; . \tag{11}$$

holds. This gives $\lim_{N\to\infty} \Delta_N = 0$. So if we consider multitone signals with arbitrary phase functions the exact asymptotic behavior of the numbers $\Delta_N, N \in I\!N$, is now known. The function P_N is shown in Figure 1.

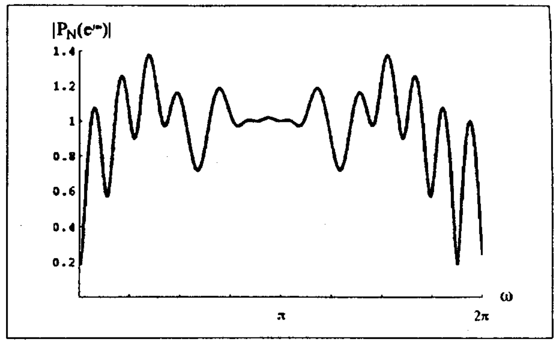

Figure 1: Function $|P_N|$ for $N = 16$

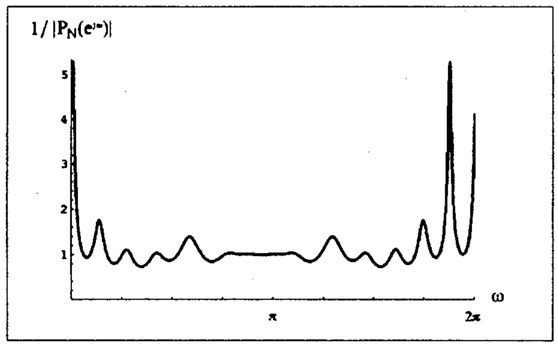

Figure 2: Inverse function $\frac{1}{|P_N|}$ for $N = 16$

3 Proof of (11)

We have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(|P_N(e^{j\omega})|^2 - 1 \right)^2 d\omega =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_N(e^{j\omega})|^4 \, d\omega - 1 \; . \tag{12}$$

For the proof of (11) it is enough to show that

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_N(e^{j\omega})|^4 \, d\omega = 1 \tag{13}$$

holds. For the proof of (13) we investigate the function $P_{*,N}(e^{j\omega}) = P_N\left(e^{j(\omega + \frac{\pi}{N+1})}\right)$.

Of course the function $P_{*,N}$ is also a multitone function. The phase function ϕ_N has the form

$\phi_{*,N}(k) = \frac{k^2\pi}{N+1}$. We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_N(e^{j\omega})|^4\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_{*,N}(e^{j\omega})|^4\, d\omega \ .$$

In the next step it is shown that there exists a positive constant C_7 such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_{*,N}(e^{j\omega})|^4\, d\omega \le 1 + \frac{C_7}{(N+1)^{\frac{2}{5}}} \qquad (14)$$

holds. For this we investigate the number

$$I_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_{*,N}(e^{i\omega})|^4\, d\omega \ . \qquad (15)$$

We use the function

$$\begin{aligned} H_N(e^{i\omega}) &= P_{*,N}(e^{i\omega}) \cdot \overline{P_{*,N}(e^{i\omega})} \\ &= \sum_{l=-N}^{N} h_N(l)\exp(-il\omega) \ . \end{aligned} \qquad (16)$$

We have with Parseval's equation

$$I_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_N(e^{i\omega})|^2\, d\omega = \sum_{l=-N}^{N} |h_N(l)|^2 \ . \qquad (17)$$

Now the coefficients h_N are calculated. We have with (16)

$$\begin{aligned} H_N(e^{i\omega}) &= \sum_{l_1=0}^{N} p_{*,N}(l_1)e^{-il_1\omega} \cdot \sum_{l_2=0}^{N} \overline{p_{*,N}(l_2)}e^{-il_2\omega} \\ &= \sum_{l=0}^{N}\sum_{k=-l}^{N-l} p_{*,N}(l)\overline{p_{*,N}(l+k)}e^{-ik\omega} \\ &= \sum_{k=-N}^{N} e^{-ik\omega}\sum_{l=0}^{N} p_{*,N}(l)\overline{p_{*,N}(l+k)}q(k,l) \end{aligned} \qquad (18)$$

The function q is defined by

$$q(k,l) = \begin{cases} 1 & -l \le k \le N-l \\ 0 & k \notin [-l, N-l] \ . \end{cases} \qquad (19)$$

This gives for the time instants $0 \le k \le N$

$$h_N(k) = \sum_{l=0}^{N-k} p_{*,N}(l)\overline{p_{*,N}(l+k)} \ . \qquad (20)$$

For $-N \le k \le -1$ we have

$$h_N(k) = \sum_{l=|k|}^{N} p_{*,N}(l)\overline{p_{*,N}(l+k)} \ . \qquad (21)$$

This gives for all time instants $0 \le k \le N$

$$|h_N(k)| = \frac{1}{N+1}\cdot\left|\frac{1-\exp\left(-i\frac{2k(N-k+1)\pi}{N+1}\right)}{1-\exp\left(-i\frac{2k\pi}{N+1}\right)}\right| \ . \qquad (22)$$

For $-N \le k \le -1$ similar investigations give

$$|h_N(k)| = \frac{1}{N+1}\cdot\left|\frac{1-\exp\left(-i\frac{2k(N-|k|+1)\pi}{N+1}\right)}{1-\exp\left(-i\frac{2k\pi}{N+1}\right)}\right| \ . \qquad (23)$$

We use the equations (22) and (23) for the calculation of the number I_N. If $k=0$ we have

$$h_N(0) = \sum_{l=0}^{N} p_{*,N}(l)\overline{p_{*,N}(l)} = 1 \ . \qquad (24)$$

This gives

$$I_N = 1 + \sum_{k=1}^{N}|h_N(k)|^2 + \sum_{k=-N}^{-1}|h_N(k)|^2 \ . \qquad (25)$$

Now we calculate the sums on the right side of (25). The first sum is denoted by S_N^1. Let k_N be the largest number such that

$$\frac{k_N^2}{N+1} \le \frac{1}{(N+1)^{\frac{1}{5}}} \qquad (26)$$

holds. Let K_N be the smallest number such that

$$(N+1) - (N+1)^{\frac{2}{5}} \le K_N \qquad (27)$$

holds. We have with these numbers

$$S_N^1 = \sum_{k=1}^{k_N}|h_N(k)|^2 + \sum_{k=k_N+1}^{K_N-1}|h_N(k)|^2 + \sum_{k=K_N}^{N}|h_N(k)|^2 \ . \qquad (28)$$

In the next step the sums on the right side of (28) are investigated. We have for $1 \le k \le k_N$

$$\left|1-\exp\left(-i\frac{2k(N-k+1)\pi}{N+1}\right)\right| \le C_2\cdot\frac{1}{(N+1)^{\frac{1}{5}}} \ . \qquad (29)$$

C_2 is a certain constant [1]. This gives for the first sum

$$\begin{aligned} \sum_{k=1}^{k_N}|h_N(k)|^2 &\le \frac{4(C_2)^2}{(N+1)^{\frac{12}{5}}}\cdot\sum_{k=1}^{k_N}\frac{1}{\left(\sin\left(\frac{k\pi}{N+1}\right)\right)^2} \\ &\le \frac{C_3}{(N+1)^{\frac{2}{5}}} \ . \end{aligned} \qquad (30)$$

Here we use for $1 \le \frac{k\pi}{N+1} \le \frac{\pi}{2}$ the inequality

$$\sin\frac{k\pi}{N+1} \ge \frac{k}{N+1} \ . \qquad (31)$$

Now the second sum on the right side of (28) is investigated. Let r_N be the largest number such that

$$\frac{r_N \pi}{N+1} \le \frac{\pi}{2} \qquad (32)$$

holds. We have with the number r_N

$$\sum_{k=k_N+1}^{K_N-1} |h_N(k)|^2 = \sum_{k=k_N+1}^{r_N} |h_N(k)|^2 + \sum_{k=r_N+1}^{K_N-1} |h_N(k)|^2 . \qquad (33)$$

The first sum on the right side of (33) can be calculated by

$$\sum_{k=k_N+1}^{r_N} |h_N(k)|^2 \le \frac{1}{k_N+1} \le \frac{2}{(N+1)^{\frac{2}{5}}} . \qquad (34)$$

Here we have used $k_N \ge (N+1)^{\frac{2}{5}} - 1 \ge \frac{1}{2}(N+1)^{\frac{2}{5}}$. The second sum on the right side of (33) can be calculated by

$$\sum_{k=r_N+1}^{K_N-1} |h_N(k)|^2 \le \frac{1}{N-K_N} \le \frac{1}{(N+1)^{\frac{2}{5}}} . \qquad (35)$$

In the next step the third sum on the right side of (28) is investigated. It can be shown that [1]

$$\sum_{k=K_N}^{N} |h_N(k)|^2 \le \frac{C_4}{(N+1)^{\frac{2}{5}}} . \qquad (36)$$

holds. This gives

$$\sum_{k=1}^{N} |h_N(k)|^2 \le \frac{C_5}{(N+1)^{\frac{2}{5}}} , \qquad (37)$$

where C_5 is a certain constant. Similar arguments show [1]

$$\sum_{k=-N}^{-1} |h_N(k)|^2 \le \frac{C_6}{(N+1)^{\frac{2}{5}}} . \qquad (38)$$

This gives

$$I_N \le 1 + \frac{C_7}{(N+1)^{\frac{2}{5}}} . \qquad (39)$$

For a complete proof see [1].

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