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An Extension of Sandberg's Representation Theorem for Linear Time-Continuous Systems

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Abstract
The input output map of linear time-continuous systems is investigated. The class of all systems for which the input output map is given by the usual integral is investigated. A complete characterization of this class of systems is given in the paper.

Keywords: Linear Systems, Modelling, BIBO-stable Systems, Signal-Processing

1 Introduction

This paper deals with an extension of a representation theorem for linear time-continuous systems recently discovered by I.W. Sandberg. It was shown by I.W. Sandberg, that if a linear time-continuous causal system satisfies certain conditions, then the representation

$$(Sx)(t) = \int_{-\infty}^{t} h(t,\tau)x(\tau)\,d\tau + \lim_{a\to-\infty}(SP_af)(t) \tag{1}$$

holds for all $t \in I\!R$. Here the function h has the usual impulse-response interpretation and the function P_ax is given by $(P_ax)(t) = x(t)$ if $t \leq a$ and $(P_ax)(t) = 0$ if $t > a$. So if the signal x is nonzero on a finite time interval $[a,b]$ only then the input-output map of the system S is given for that signal by

$$(Sx)(t) = \int_{a}^{b} h(t,\tau)x(\tau)\,d\tau\ . \tag{2}$$

The main idea of the theory of time-continuous single-input single-output linear systems is that every such system S has an input-output map that can be represented by

$$(Sx)(t) = \int_{-\infty}^{\infty} h(t,\tau)x(\tau)\,d\tau\ . \tag{3}$$

The system S need not be a causal system. Almost always it is emphasized that the representation (3) holds for all time-continuous linear systems [5] [6]. It was recently discovered that such a representation does not hold for all time-continuous single-input single-output linear systems S. The first counter examples were constructed in [3] [8].

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2 Sandberg's Condition

At first we need some preliminaries. Let $I\!R$ be the set of real numbers, and let $L^\infty(I\!R)$ denote the normed signal space of essentialy bounded real valued Lebesgue measurable functions x. The norm is given by

$$\|x\| = esssup_{t\in \mathbf{R}} |x(t)| \ . \tag{4}$$

A linear system is a linear map of $L^\infty(I\!R)$ into itself. A linear system S is called causal if the equation $P_a S = P_a S P_a$ holds for all $a \in I\!R$. We refer to the following two conditions as Sandberg's conditions:

A1) For each t and $a \in I\!R$ with $a < t$, there is a real constant $c_{t,a}$ such that

$$|(Sx)(t)| \leq c_{t,a} \cdot \int_a^t |x(\tau)| \, d\tau \tag{5}$$

holds for all $x \in L^\infty(I\!R)$ with $x(\tau) = 0$ for $\tau < a$.

A2) The inequality $\sup_{a\in \mathbf{R}} |(SP_a x)(t)| < \infty$ holds for each $x \in L^\infty(I\!R)$ and $t \in I\!R$.

Let $\tau \in I\!R$ be an arbitrary number. We consider for $\delta > 0$ the signal

$$w_{\tau,\delta}(\tau_1) = \begin{cases} \frac{1}{\delta} & \tau_1 \in [\tau, \tau+\delta) \\ 0 & \tau_1 \notin [\tau, \tau+\delta) \ . \end{cases} \tag{6}$$

The following theorem was shown in [8].

Theorem 1 (Sandberg) *Let S be a causal system that meats Sandberg's conditions A1) and A2). Then the following holds for all $t \in I\!R$ and all signals $x \in L^\infty(I\!R)$.*
i) The function

$$h(t,\tau) = \lim_{\delta\to 0} (S w_{\tau,\delta})(t) \tag{7}$$

exists for allmost all $\tau \in I\!R$.
ii) The system S has the representation

$$(Sx)(t) = \int_{-\infty}^t h(t,\tau) x(\tau) \, d\tau + \lim_{a\to-\infty} (SP_a x)(t) \ . \tag{8}$$

Similar results for time-discrete linear systems were obtained in [1] and [2].
At next we want to give an extension of Sandberg's representation theorem. Let p be a positive number which satisfies $1 \leq p < \infty$. We say that a system S satisfies the condition A(p) if the following holds.
For each t and $a \in I\!R$ with $a < t$, there is a real constant $c_{t,a}(p)$ such that

$$|(Sx)(t)| \leq c_{t,a}(p) \cdot \left(\int_a^t |x(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \tag{9}$$

holds for all $x \in L^\infty(I\!R)$ with $x(\tau) = 0$ for $\tau < a$.

At first we have to show that the condition A(p) is weaker than the condition A1), that means if the system S satisfies the condition A1) then it also satisfies the condition A(p) for all $p > 1$.

For this let $a < t$ be an abitriary number and let $x \in L^\infty(I\!R)$ be a signal with $x(\tau) = 0$ for $\tau < a$. Then we have for $p > 1$

$$\begin{aligned}
\int_a^t |x(\tau)|\, d\tau &= \int_a^t |x(\tau)| \cdot 1 \cdot\, d\tau \\
&\leq \left(\int_a^t |x(\tau)|^p\, d\tau \right)^{\frac{1}{p}} \cdot \left(\int_a^t 1\, d\tau \right)^{\frac{1}{q}} \\
&= (t-a)^{\frac{1}{q}} \cdot \left(\int_a^t |x(\tau)|^p\, d\tau \right)^{\frac{1}{p}} . \qquad (10)
\end{aligned}$$

(Here we have used Hölders inequality with $\frac{1}{p} + \frac{1}{q} = 1$.) That means if the system S satisfies the condition A1) with the constant $c_{t,a}$ than it satisfies also the condition A(p) with the constant $(t-a)^{\frac{1}{q}} \cdot c_{t,a}$. In the paper the following result is proved.

Theorem 2 *Let S be an abitriary causal system such that the conditions $A(p)$ for a $p > 1$ and A2) are met. Then the following holds:*
i) The function

$$h(t,\tau) = \lim_{\delta \to 0} (S w_{\tau,\delta})(t) \qquad (11)$$

exists for allmost all $\tau \in I\!R$ and satisfies

$$\left(\int_a^t |h(t,\tau)|^q\, d\tau \right)^{\frac{1}{q}} < \infty \qquad (12)$$

for each $a \in I\!R$, where q is given by $\frac{1}{p} + \frac{1}{q} = 1$ [7].
ii) The input-output map of the system S has the form (8).

Next, BIBO-stable (bounded input-bounded output) linear systems are investigated. This means that a positive number C_1 exists, such that for all signals $x \in L^\infty(I\!R)$ the inequality

$$\|Sx\| \leq C_1 \cdot \|x\| \qquad (13)$$

holds.
Since we have $|(SP_a x)(t)| \leq C_1 \cdot \|P_a x\| \leq C_1 \|x\|$, all BIBO-stable systems satisfy the condition A2). With the Theorem 2 we get the following Corollary.

Corollary 1 *Let S be a causal BIBO-stable linear system such that the condition $A(p)$ is met for some $p \geq 1$. Then the following representation holds for all signals $x \in L^\infty(I\!R)$*

$$(Sx)(t) = (S_1 x)(t) + (S_\infty x)(t) \; . \qquad (14)$$

The system S_1 has the form

$$(S_1 x)(t) = \int_{-\infty}^t h(t,\tau) x(\tau)\, d\tau \; . \qquad (15)$$

The function $h(t,\tau)$ in (15) is given by $h(t,\tau) = \lim_{\delta \to 0} (S w_{\tau,\delta})(t)$, which exists for almost all $\tau \in I\!R$.

The system S_∞ is defined by

$$(S_\infty x)(t) = \lim_{a \to \infty} (SP_a x)(t) \ . \tag{16}$$

Both systems S_1 and S_∞ are causal BIBO-stable linear systems.

The limit in (16) exists for all signals $x \in L^\infty(I\!R)$. The representation (14) means that each causal BIBO-stable linear system S which satisfies the condition A(p) can be split into two subsystems S_1 and S_∞ which are given by (15) and (16). The input-output map of the system S_1 is given by the usual integral. The input-output map of the system S_∞ don't has this form. For all signals x which are non zero on a finite time interval only, we have

$$(S_\infty x)(t) = \lim_{a \to \infty} (SP_a x)(t) = 0 \tag{17}$$

for all $t \in I\!R$. That means the impulse-response of the system S_∞ is zero for all $t \in I\!R$. So the system S and S_1 have the same impulse-response.

3 Strongly Continuous Systems

At the end of this paper we investigate those causal BIBO-stable linear systems S, that are characterized by their impulse response. Of course we suppose, that such a system satisfies condition $A(p)$ for some $p \geq 1$. That means to find all causal BIBO-stable linear systems, so that the indentity $S = S_1$ holds. For this reason it is necessary to introduce one more concept. A sequence of input signals $x_n \in L^\infty(I\!R)$ is said to be convergent to a signal $x \in L^\infty(I\!R)$, if a positive number C_1 exists, so that

$$\|x_n\| \leq C_1 \qquad , n \in I\!N \, , \tag{18}$$

and

$$\lim_{n \to \infty} x_n(t) = x(t) \tag{19}$$

hold for all $t \in I\!R$.
A causal BIBO-stable linear system S is said to be a strongly continuous system, if for each convergent sequence of signals $x_n \in L^\infty(I\!R)$ the equation

$$(Sx)(t) = \lim_{n \to \infty} (Sx_n)(t) \tag{20}$$

holds.
Each norm-convergent sequence of signals $x_n \in L^\infty(I\!R)$ is also convergent but not vice-versa. This means the concept of convergence is weaker than the concept of norm-convergence. With these consepts we get the following theorem.

Theorem 3 *Let S be a causal BIBO-stable linear system such that condition $A(p)$ is met for some $p \geq 1$. Then the system S has the form*

$$(Sx)(t) = \int_{-\infty}^{t} h(t,\tau)x(\tau)\, d\tau \tag{21}$$

if and only if the system S is strongly continuous.

Theorem (3) characterizes all causal BIBO-stable systems, for which the input-output map is given by the usual integral. Of course the systems have to satisfy condition A(p) for some $p \geq 1$. It can also be shown that the system S in Theorem (3) need not be a causal system. These is a consequence of the following Corollary.

Corollary 2 *Let S be a BIBO-stable strongly continuous linear system such that condition $A(p)$ is met for some $p \geq 1$. Then the system S is also causal.*

4 Proof of Theorem 3

In this section a proof of the Theorem 3 is given.
Let S be an arbitrary causal BIBO-stable linear system such that condition A(p) is met for some $p \geq 1$. Suppose that the system S is strongly continuous. Let $x \in L^\infty(\mathbb{R})$ be an arbitrary input signal. For each $n \in \mathbb{N}$ we consider the signal

$$x_n(t) = \begin{cases} x(t) & |t| \leq n \\ 0 & |t| > n \ . \end{cases}$$

We have

$$(Sx_n)(t) = \int_{-n}^{n} h(t,\tau)x(\tau)\,d\tau \ . \tag{22}$$

Since the system S is strongly continuous we have

$$\begin{aligned} (Sx)(t) &= \lim_{n\to\infty}(Sx_n)(t) = \lim_{n\to\infty}\int_{-n}^{n} h(t,\tau)x(\tau)\,d\tau \\ &= \int_{-\infty}^{t} h(t,\tau)x(\tau)\,d\tau \ . \end{aligned} \tag{23}$$

So we have equation (21) for all input signals.

Now let x_n be an arbitrary convergent sequence. Suppose that the input output map of the system S is given by (21). Then we have

$$\begin{aligned} \lim_{n\to\infty}(Sx_n)(t) &= \lim_{n\to\infty}\int_{-\infty}^{t} h(t,\tau)x_n(\tau)\,d\tau \\ &= \int_{-\infty}^{t} h(t,\tau)x(\tau)\,d\tau = (Sx)(t) \ , \end{aligned} \tag{24}$$

so the system S is strongly continuous. This proofs the Theorem 3.

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