ON THE CONVEXITY OF REACHABLE SETS OF NONLINEAR DYNAMIC SYSTEMS – AN IMPORTANT STEP IN GENERATING DISCRETE ABSTRACTIONS OF CONTINUOUS SYSTEMS

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Abstract. Given a flow $\varphi\colon U \subseteq T \times X \to X$ of some ordinary differential equation $\dot{x} = f(x) \quad (*)$, subsets $\Omega_1, \Omega_2 \subseteq X$ of the state space, and some $\tau > 0$ with $[0, \tau] \times \Omega_1 \subseteq U$, we investigate whether $\varphi(\tau, \Omega_1)$ and Ω_2 intersect. In other words, we examine, whether Ω_2 is reachable from Ω_1 after elapsed time τ. This problem plays a central part in a recently proposed approach to systems analysis via discrete abstractions, where Ω_1 and Ω_2 are members of a covering of the state space of $(*)$. We propose to view this problem as an optimisation problem, which is convex iff Ω_1 and Ω_2 are so. We derive conditions under which the image of a set under some C^∞-diffeomorphism is convex and show that these results apply to the time-τ-map $\varphi(\tau, \cdot)$ of $(*)$ under mild conditions on f. We further show that the image of a ball under the time-τ-map is convex provided its radius is sufficiently small and give an upper bound on that radius depending on τ and the first and second derivatives of f only. This shows that the approach proposed in this paper, namely, to treat the reachability problem arising in the context of discrete abstractions as an optimisation problem, applies to virtually any differential equation $(*)$ and yields convex problems provided that the state space is covered by sufficiently small balls.

Key Words. Reachable sets, reachability, discrete abstractions, convexity.

1. INTRODUCTION

The problem of whether a family of solutions of a dynamic system pass through a prescribed subset of the state space arises in a variety of applications. A simple instance of that problem is the following: given a flow $\varphi\colon U \subseteq T \times X \to X$, subsets $\Omega_1, \Omega_2 \subseteq X$ of the state space, and some $\tau > 0$ with $[0, \tau] \times \Omega_1 \subseteq U$, determine whether

$$\varphi(\tau, \Omega_1) \cap \Omega_2 \neq \emptyset, \tag{1}$$

that is, whether Ω_2 *is reachable from* Ω_1 *after elapsed time* τ. (See Fig. 1.)
In the following, a particular application, which motivated the work of the authors of this paper, is described.

In a recently proposed approach to analysis, synthesis and verification of systems, *discrete abstractions* of the system are investigated, rather than its continuous or hybrid dynamics itself [3, 5, 8, 9]. A discrete abstraction can be represented by a finite, in general non-deterministic automaton which is based on a covering of the state space by a family of sets called *cells*. In the simplest case, those cells correspond to states of the automaton, and a transition from one state to a second is possible if the cell corresponding to the second state is reachable in the original dynamic system from the one corresponding to the first state. In other words, a reachability problem is to be solved for each pair of states, and the solution of those reachability problems is the central and most complex subtask in the generation of discrete abstractions.

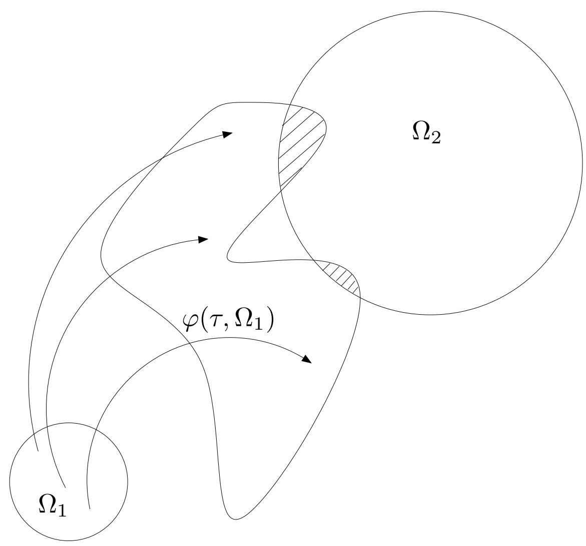

Fig. 1 Illustration of reachability problem (1).

For systems determined by ordinary differential equations

$$\dot{x} = f(x), \quad f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n, \tag{2}$$

$U \subseteq \mathbb{R}^n$ open, the reachability problem is efficiently solvable up to an arbitrarily prescribed precision if f is linear. In contrast, known approaches for nonlinear equations (2) usually are limited to rather restricted classes of right hand sides f [7], create prohibitively complex computational algorithms or are limited in accuracy [2, 6, 10]. In particular, it has been proposed to approximate the set $\varphi(\tau, \Omega_1)$ in (1) by polyhedrals [2], ellipsoids [6], and oriented rectangular hulls [10]. These approximations lead to optimisation problems that are in general non-convex and extremely complex to solve with high accuracy.

In this paper we propose to treat the reachability problem (1) directly, without approximating $\varphi(\tau, \Omega_1)$. The idea is that the validity of (1) can be efficiently checked up to an arbitrary precision if both Ω_2 and $\varphi(\tau, \Omega_1)$ are convex. In fact, the reachability problem becomes a very special case of a convex optimisation problem. As Ω_2 may be chosen convex, the question arises under what conditions $\varphi(\tau, \Omega_1)$ is convex.

In Section 2., we derive conditions under which the image of a set under some C^∞ diffeomorphism is convex. Furthermore, we give a necessary and sufficient condition on the radius of the ball and the first and second derivatives of the diffeomorphism for the image of the ball to be convex. From that condition it follows that the image of a ball is convex provided its radius is sufficiently small, and upper bounds on the radius can be obtained.

In Section 3. we apply these results to the time-τ-map of the autonomous differential equation (2). We show that the results from Section 2. apply to the time-τ-map of (2) under mild conditions on f. In particular, $\varphi(\tau, \Omega_1)$ will be convex if Ω_1 is a sufficiently small ball. This shows that, in principle, the approach proposed in this paper, namely, to treat (1) directly, applies to virtually any differential equation (2) provided that the state space is covered by sufficiently small balls.

2. THE CONVEXITY OF IMAGE SETS UNDER DIFFEOMORPHISMS

In this section we investigate the convexity of image sets under C^∞ diffeomorphisms. Firstly, we give a sufficient condition for sets to be convex in general and from that we derive a second theorem which specifies the conditions for convex image sets. Subsequently we consider specifically image sets of balls.

Theorem 1 gives the conditions for the convexity of a set described by a map k.

Theorem 1 *Let $U \subseteq \mathbb{R}^n$ be open, the map $k \colon U \to \mathbb{R}$ be a C^∞ diffeomorphism, the set $\Omega = \{ x \in U \mid k(x) \leq 0 \}$ be compact, its boundary $\partial\Omega$ be connected and $k'(x) \neq 0$ for all $x \in \partial\Omega$.*
If the quadratic form

$$\ker k'(x) \to \mathbb{R} \colon h \mapsto k''(x)h^2 \tag{3}$$

is positive definite for all $x \in \partial\Omega$ then the set Ω is convex.

Proof A proof can be found in [11]. ∎

This theorem is strongly related to the terms curvature and second fundamental form in differential geometry. One can show that the quadratic form (3) is equal to the second fundamental form at a point x on the boundary of a set up to a positive scalar factor. The second fundamental form is used to assess the curvature of a surface at a single point. The relation of the quadratic form (3) to the curvature at an arbitrary point x_0 on the boundary of a set Ω is shown in Fig. 2. If the curvature is positive for all points of a surface then the surface is convex. Thus, we can derive the global property, the convexity of a surface, from the local property of positive definiteness at all points. Note further that condition (3) does not depend on k but on Ω only.

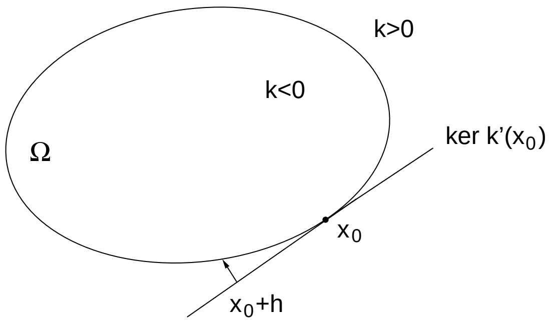

Fig. 2 If the quadratic form (3) of Theorem 1 is positive definite at x_0 and at all other points on the boundary of Ω, then Ω is convex.

Theorem 1 allows us to assess the convexity of a set if we know the quadratic form (3) of this set. For the analysis of reachable sets this theorem alone would not be applicable because reachable sets are image sets about which we generally do not know the quadratic form (3) explicitly. On the other hand,we have some information about the initial set and the flow of the differential equation. Therefore, let us consider an image set of Ω under a map F, where F is assumed to be a C^∞ diffeomorphism, and specify the conditions for the convexity of the image set $F(\Omega)$.

Theorem 2 *Let $U \subseteq \mathbb{R}^n$ be open, $g\colon U \to \mathbb{R} \in C^\infty$, $\Omega = \{ x \in U \mid g(x) \le 0 \}$ be compact, its boundary $\partial\Omega$ connected and $g'(x) \neq 0$ for all $x \in \partial\Omega$.*
Let further $V \subseteq \mathbb{R}^n$ be open and the map $F\colon U \to V$ be a C^∞ diffeomorphism. If the quadratic form

$$\ker g'(x) \to \mathbb{R}\colon \quad h \mapsto g''(x)h^2 - g'(x)F'(x)^{-1}F''(x)h^2 \tag{4}$$

is positive definite for all $x \in \partial\Omega$, then the set $F(\Omega)$ is convex.

Proof We know that $F(\partial\Omega) = \partial F(\Omega)$, the set $F(\Omega)$ is compact and the boundary $\partial F(\Omega)$ is connected because of the continuity of F and F^{-1}.
1. From $F \circ F^{-1} = \mathrm{id}_V$ we obtain

$$F'(F^{-1}(y))(F^{-1})'(y)h = h \tag{5}$$

for all $y \in V$ and all $h \in \mathbb{R}^n$. We differentiate Equation (5)

$$F''(F^{-1}(y))F'(F^{-1})h(F^{-1})'(y)h + F'(F^{-1}(y))(F^{-1})''(y)h^2 = 0$$

and apply Equation (5) to its derivation:

$$F''(F^{-1}(y))\left((F^{-1})'(y)h\right)^2 + F'(F^{-1}(y))(F^{-1})''(y)h^2 = 0.$$

Thus, we yield following equation:

$$(F^{-1})''(y)h^2 = -F'(F^{-1}(y))^{-1}F''(F^{-1}(y)) \left(F'(F^{-1}(y))^{-1}h\right)^2. \tag{6}$$

2. We now consider the map $k = g \circ F^{-1}$ defined on V, where k describes the image set $F(\Omega)$ as shown in Fig. 3.

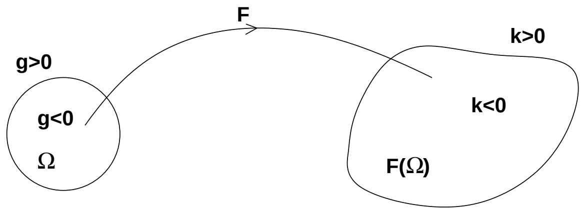

Fig. 3 The set Ω is described by g and its image set $F(\Omega)$ is described by $k = g \circ F^{-1}$.

It is obvious that $F(\Omega) = \{ y \in V \mid k(y) \le 0 \}$ because $y \in F(\Omega)$ and $F^{-1}(y) \in \Omega$ are equivalent. Furthermore, the conditions on g and F imply that k is a C^∞ diffeomorphism and is therefore differentiable. Hence, we obtain

$$k'(y) = g'(F^{-1}(y))(F^{-1})'(y) = g'(F^{-1}(y))F'(F^{-1}(y))^{-1}$$

for all $y \in V$. It follows that $k'(y) \neq 0$ for all $y \in \partial F(\Omega)$ due to the bijectivity of $F'(F^{-1}(y))$ and we can also conclude that

$$\ker k'(y) = F'(F^{-1}(y)) \ker g'(F^{-1}(y)) \tag{7}$$

because of following equivalences for $h \in \ker k'$ and $h \in \mathbb{R}^n$:

$$\begin{aligned} h \in \ker k' &\Leftrightarrow k'h = 0 \\ &\Leftrightarrow g'(F^{-1}(y))F'(F^{-1}(y))^{-1}h = 0 \\ &\Leftrightarrow F'(F^{-1}(y))^{-1}h \in \ker g'(F^{-1}(y)) \\ &\Leftrightarrow h \in F'(F^{-1}(y)) \ker g'(F^{-1}(y)). \end{aligned}$$

The differentiation of k' results in

$$k''(y)h^2 = g''(F^{-1}(y))\left((F^{-1})'(y)h\right)^2 + g'(F^{-1}(y))(F^{-1})''(y)h^2$$

for all $h \in \mathbb{R}^n$. By applying (6) and denoting $x = F^{-1}(y)$ we obtain:

$$k''(y)h^2 = g''(x)\left(F'(x)^{-1}h\right)^2 - g'(x)F'(x)^{-1}F''(x)\left(F'(x)^{-1}h\right)^2. \tag{8}$$

3. To apply Theorem 1 we have to show that the quadratic form

$$F'(x)\ker g'(x) \to \mathbb{R}\colon h \mapsto k''(y)h^2 \tag{9}$$

is positive definite for all $y = F(x) \in \partial F(\Omega)$. This is with the bijectivity of $F'(x)$ equivalent to the positive definiteness of

$$\ker g'(x) \to \mathbb{R}\colon h \mapsto k''(y)\,(F'(x)h)^2 .$$

With (8) we obtain

$$k''(y)\,(F'(x)h)^2 = g''(x)h^2 - g'(x)F'(x)^{-1}F''(x)h^2,$$

i.e. that Equation (9) is positive definite if Equation (4) is positive definite. ∎

Theorem 2 puts us in the position to assess the convexity of an image set without knowing the image set exactly. We just need some information on the properties of the set Ω and the map F to calculate the quadratic form (4).

It is "attractive" to consider the image sets of balls because balls are naturally convex and their curvature can be determined by the radius, i.e. we can influence the quadratic form (4) by the radius of the ball which is expressed in the following corollary.

Corollary 3 *Let $x_0 \in \mathbb{R}^n$ and Ω be a closed ball centered at x_0. Let further $U, V \subseteq \mathbb{R}^n$ be open, $\Omega \subseteq U$ and $F\colon U \to V$ be a C^∞ diffeomorphism.*
The set $F(\Omega)$ is convex if

$$\left\langle x - x_0 \,\middle|\, F'(x)^{-1}F''(x)h^2 \right\rangle < 1 \tag{10}$$

for all $x \in \partial\Omega$ and all $h \in \operatorname{span}\{x - x_0\}^\perp$ with $\|h\|_2 = 1$, where $\|\cdot\|_2$ denotes the Euclidean norm.

Proof We define g on $\mathbb{R}^n$ by

$$g(x) = \|x - x_0\|_2^2 - r^2 \tag{11}$$

for a radius $r > 0$ such that Ω and g fulfill the conditions of Theorem 2. We can show that F fulfills the conditions of Theorem 2 as well. So, we need to determine $g'(x)$, $\ker g'(x)$ and $g''(x)$ which results in $g'(x)h = 2\langle x - x_0 | h\rangle$, $\ker g'(x) = \operatorname{span}\{x - x_0\}^\perp$ and $g''(x)h^2 = 2\|h\|_2^2$. After insertion in the quadratic form (4) we obtain:

$$\operatorname{span}\{x - x_0\}^\perp \to \mathbb{R}\colon\quad h \mapsto 2\|h\|_2^2 - 2\langle x - x_0 | F'(x)^{-1}F''(x)h^2\rangle. \tag{12}$$

The quadratic form (12) is positive definite iff

$$0 < 1 - \left\langle x - x_0 | F'(x)^{-1}F''(x)h^2 \right\rangle \tag{13}$$

holds for all $h \in \operatorname{span}\{x - x_0\}^\perp$ with $\|h\|_2 = 1$, which is equivalent to condition (10). ∎

Corollary 3 says when the image of a ball Ω under a C^∞ diffeomorphism F is convex. We can see that the convexity of the image set depends on the radius of the ball and the first and second derivatives of F. We can follow that if the radius of the ball is chosen sufficiently small for given F' and F'' its image set becomes convex under the conditions of Corollary 3. Using an estimate we can give a sufficient condition for the convexity of an image set: if the radius of Ω is smaller than $1/\left\|F'(x)^{-1}F''(x)h^2\right\|_2$ or the upper bound of the radius is

$$\frac{1}{\|F'(x)^{-1}\|_2\;\|F''(x)h^2\|_2} \tag{14}$$

respectively for all $x \in \partial\Omega$ and all $h \in \operatorname{span}\{x - x_0\}^\perp$ with $\|h\|_2 = 1$, then the image set $F(\Omega)$ is guaranteed to be convex.

3. REACHABLE SETS OF CONTINUOUS SYSTEMS

We apply the results on the convexity of image sets from the previous section to the reachability problem (1) by investigating image sets under the flow of a differential equation. In particular, we derive a condition for the image set $\varphi(\tau, \Omega_1)$ to be convex, which turns the reachability problem into a convex optimisation problem as explained in the introductory section of this paper.

In the following, we refer to a system described by ordinary autonomous differential equations (2). The consideration of autonomous first-order systems only means no loss of generality because most differential equations can be transformed into differential equations (2) for our application.

Now, we consider the map F of Theorem 2 and Corollary 3 as the time-τ-map $\varphi(\tau, \cdot)$ of (2). The time-τ map represents the impact of the flow of system (2) on initial values x_0 for an elapsed time τ and is a C^∞ diffeomorphism if the right hand side f of (2) is a C^∞ diffeomorphism [1].

In general, for a nonlinear system, the image set of a convex set under the time-τ-map is non-convex.To apply Corollary 3 we further assume the set $\Omega_1 \subset X$ to be a ball. We can then influence the convexity via the radius of the initial sets or by changing the properties of time-τ-map. In Section 2. we derived the upper bound (14) for the radius of balls which guarantees the image sets to be convex for a given map F. Fig. 4 illustrates how convexity of image sets depends on the radius of balls: by reducing the latter beyond a certain bound, image sets eventually become convex. We now ask the question, for which systems and under which conditions does such a bound exist.

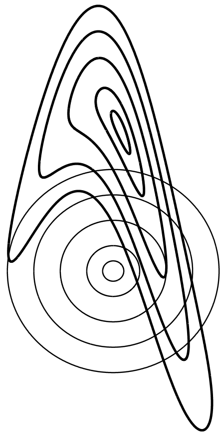

Fig. 4 Image sets (bold) of circular sets of a two-dimensional nonlinear example system in the phase plane. Reducing the radius of the circular sets eventually results in convex image sets.

We can show that such a bound exists for a wide class of systems using the following approximations based on Gronwall's lemma for the norm of the first and second derivative of the solution of a differential equation [4]:

Theorem 4 *Let $f : U \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be twice continuously differentiable and φ the general solution of $\dot{x} = f(t,x)$ and $\forall_{(t,x)\in U}\forall_{h\epsilon\mathbb{R}^n}$ $\|D_2 f(t,x)\| \leq L_1$ and $\|D_2^2 f(t,x)h^2\| \leq L_2\|h\|^2$ for real constants L_1 and L_2 with $L_1 \neq 0$, where $D_i^j f$ denotes the jth partial derivative of f with respect to its ith argument. Then the inequalities*

$$\|D_3\varphi(t,t_0,x_0)\| \leq e^{L_1|t-t_0|} \tag{15}$$

and

$$\|D_3^2\varphi(t,t_0,x_0)h^2\| \leq \frac{L_2}{L_1}e^{2L_1|t-t_0|}\left(e^{L_1|t-t_0|}-1\right)\|h\|^2 \tag{16}$$

hold for all $(t,t_0,x_0) \in \mathrm{dom}\varphi$ and all $h \in \mathbb{R}^n$.

By restricting equations (15) and (16) to the autonomous case we obtain estimates for the norms of the derivatives of the time-τ-map $F'(x)$ and $F''(x)$. These estimates are in principle applicable to virtually any differential equation for which the constants L_1 and L_2 exist.

We applied (15) and (16) to an example and compared it to numerical computations of $\left\|F'(x)^{-1}\right\|_2$ and $\left\|F''(x)h^2\right\|_2$. The latter indicated, however, that the estimates obtained from (15) and (16) are very conservative and therefore not useful in practice [4].

For determining the desired bounds for the radius more accurately, we need more accurate estimates of the derivatives of the time-τ map. We think that better estimates can only be obtained for specific systems or types of systems.

Furthermore, the estimates (15) and (16) show as well that $\left\|F'(x)^{-1}\right\|_2$ and $\left\|F''(x)h^2\right\|_2$, i.e. the upper bound of the radius, depends on τ. The refinement of time discretisation consequently reduces the upper bound. This is illustrated for an example in Fig. 5.

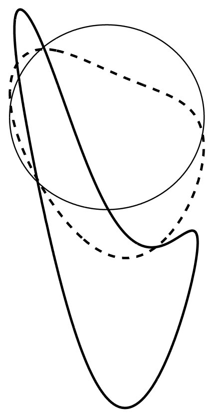

Fig. 5 Image sets (bold) of a circular set of an arbitrary two-dimensional nonlinear system in the phase plane for different values of τ. Reducing τ eventually results in convex image sets.

In this section we reduced the formulation of the reachability problem (1) as a convex optimisation problem to finding tight estimates for the Euclidean norms of the derivatives of the time-τ-map.

4. CONCLUSIONS

Reachability represents an important problem in abstraction-based approaches analysis, synthesis and verification of hybrid systems.

In the present paper, we proposed a new approach to investigate the reachability problem by formulating it as an optimisation problem. Because convexity plays an important role in opimisation techniques we focused on the formulation of the reachablity problem as a convex optimisation problem, with the convexity of image sets being the central subproblem to be solved.

We derived conditions for the convexity of image sets. In particular, for a sufficiently small ball, the image set under the time-τ-map of an autonomous differential equation is convex if some mild conditions on the right hand side f hold. The problem of providing a bound for the radius if balls was reduced to finding bounds for the Euclidean norms of derivatives of the time-τ-map. We provided a general estimate for these norms, which indicated the wide applicability of our approach. However these estimates are not tight enough in practice, hence further investigations are necessary to find better estimates for specific systems or classes of systems.

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