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Convexity of reachable sets of nonlinear discrete-time systems

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***Abstract*— We present necessary and sufficient conditions for reachable sets of discrete-time systems $x(k+1) = F(k, x(k))$ to be convex. In particular, the set of states reachable at a given time from a sufficiently small ellipsoid of initial states is always convex if F is smooth enough, and we provide explicit bounds on the size of those ellipsoids. Our results imply that outer discrete approximations with approximation depth exceeding 1 can be readily computed up to arbitrary precision. A further potential application is outer polyhedral approximation of reachable sets, which becomes almost universally applicable if those sets are known to be convex.**

Key Words. Reachability analysis, nonlinear difference equations, discrete-time systems.

I. INTRODUCTION

Reachability problems play a central part in a wide range of control related problems, including safety and liveness verification, diagnosis, controller synthesis, optimization and others [1]–[7]. In these contexts, the following reachability problem always occurs, in one form or another: Given an autonomous ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

with smooth flow $\varphi\colon U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, subsets $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ of states, and some $T \in \mathbb{R}$ with $\{T\} \times \Omega_1 \subseteq U$, determine whether

$$\emptyset \neq \varphi(T, \Omega_1) \cap \Omega_2, \tag{2}$$

that is, whether there is a state in Ω_2 that is reachable from an initial state in Ω_1 at time T. (See Fig. 1.) Condition (2) can often be efficiently verified up to arbitrary precision if both the target set Ω_2 and the reachable set $\varphi(T, \Omega_1)$ are convex [8].

Therefore, the question arises under what conditions the reachable set $\varphi(T, \Omega_1)$ is convex. As there are other applications as well in which one can take advantage of the convexity of reachable sets, the question of when those sets actually are convex has been studied for continuous-time systems

$$\dot{x} = f(t, x) \tag{3}$$

in recent years, e.g. [8]–[18]. However, despite the fact that the analogous problem for discrete-time systems

$$x(k + 1) = F(k, x(k)) \tag{4}$$

has analogous applications, it does not seem to have drawn any attention so far.

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In this paper, we present necessary and sufficient conditions for reachable sets of the discrete-time system (4) to be convex. First, we would like to discuss two potential applications of our results briefly:

Outer polyhedral approximation of reachable sets. The method of outer polyhedral approximation of reachable sets is known to apply to nonlinear continuous-time systems with inputs and disturbances under some restrictions on the nonlinearity, e.g. [19]–[21]. The point we would like to make here is that the method actually applies to any discrete-time system (4) provided that the right hand side of (4) is sufficiently smooth and the reachable set is known to be convex. In particular, our results in Section III show that the method of outer polyhedral approximation of reachable sets applies to any system (4) with twice continuously differentiable right hand side if the ellipsoid of initial states is sufficiently small.

In the setting of the present paper, and under conditions that ensure convexity of the reachable set $\varphi(k, k_0, \bar{B}(x_0, r))$, it is extremely easy to obtain outer polyhedral approximations. (See Fig. 2. $\bar{B}(x_0, r)$ denotes an ellipsoid, i.e., a closed ball of radius r centered at x_0 with respect to some inner product, and φ, the general solution of (4). See Section II for notation and terminology.) All it takes is to solve (4) with w different initial values x from the boundary $\partial B(x_0, r)$ of $\bar{B}(x_0, r)$ in a first step to obtain w points on the boundary of the reachable set. In a second step, one solves w adjoint problems

$$z(k + 1) = (D_2 F(k, \varphi(k, k_0, x)^*)^{-1} z(k), \tag{5a}$$

$$z(k_0) = v_0 \tag{5b}$$

with v_0 being an outside normal to $\bar{B}(x_0, r)$ at $x \in \partial B(x_0, r)$. (Here, $D_2 F$ denotes the partial derivative of the map F with respect to its second argument, and A^*, the adjoint to the linear map A with respect to the inner product chosen; see Section II.) By well-known properties of the adjoint equation (5a), that second step yields w outside normals to the reachable set at the w boundary points computed in the first step. Consequently, approximating a convex reachable set by means of w supporting hyperplanes requires the solution of just $2w$ discrete-time systems in $\mathbb{R}^n$.

Discrete abstractions w. approximation depth exceeding **1.** As an approach to analysis, synthesis and verification of systems, it has been proposed to investigate discrete abstractions of continuous- and discrete-time systems

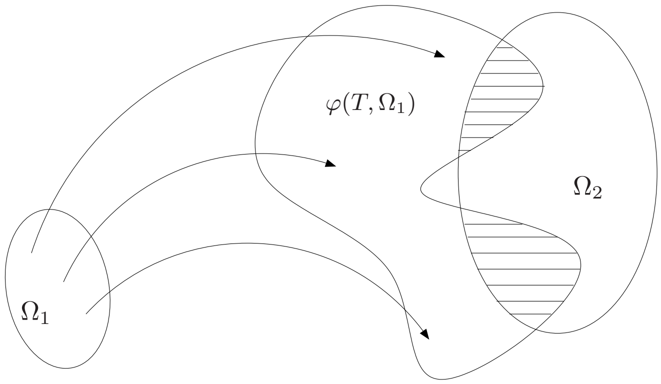

Fig. 1. Illustration of reachability problem. (See Section I.)

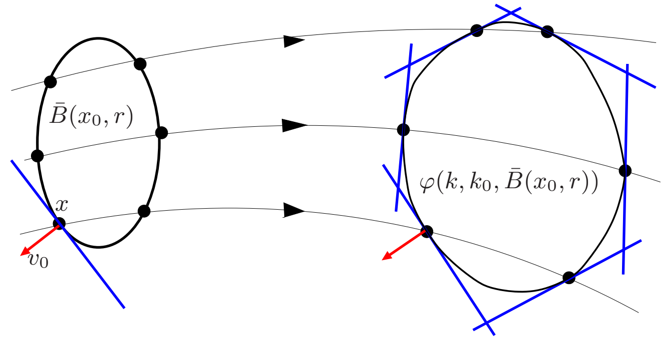

Fig. 2. Outer polyhedral approximation of convex reachable sets. (See Section I.)

rather than the dynamics of these systems itself, e.g. [1]–[4]. A discrete abstraction may be seen as a finite, in general non-deterministic automaton. In the simplest case, states of that automaton correspond to members ("cells") of a finite covering of the state space of the system. In that automaton, there is a transition from one state to another if and only if the cell corresponding to the second state is reachable from the cell corresponding to the first state via solutions of the system to be abstracted. The latter condition is exactly the reachability condition (2) if the system to be abstracted is the autonomous ODE (1). The same strategy applies to both continuous- and discrete-times systems with control inputs, $\dot{x} = f(x, u)$ or $x(k+1) = f(x(k), u(k))$, if the number of admissible control signals (in the continuous-time case: on some interval) is finite. In fact, these two cases reduce to abstractions of a finite number of nonautonomous systems (3) and (4), respectively.

We say the discrete abstraction described above is of *approximation depth* 1, since we have looked back just one step in time when constructing it. A problem with that simple kind of abstraction is that it may very well happen that there is no controller meeting the specification for the automaton. Roughly speaking, there are just too many transitions in it or, in other words, abstractions of approximation depth 1 often represent behaviors that are too rich.

It has been proposed to use discrete abstractions of *approximation depth* $l > 1$ that are constructed by looking back $l > 1$ steps in time in order to "shrink" the behavior represented by the automaton [22], [23]. Just as with abstractions of approximation depth 1, abstractions of arbitrary approximation depth l of systems with control inputs reduce to abstractions of approximation depth l of a finite number of nonautonomous systems (3) and (4), respectively.

If an abstraction with approximation depth l of the discrete-time system (4) is to be computed, transitions in a finite automaton need to be determined, where the following reachability condition is to be verified instead of (2):

$$\emptyset \neq \bigcap_{i=1}^{l+1} \varphi(k_0 + l, k_0 + i - 1, \Omega_i). \tag{6}$$

Here, k_0 is a given initial time and $(\Omega_i)_{i \in \{1,\ldots,l+1\}}$, a given family of "cells" $\Omega_i \subseteq \mathbb{R}^n$, and φ, the general solution of (4).

It should be obvious that neither (2) nor (6) can be decided precisely. Instead, the simpler condition (2) may be verified approximately, up to arbitrary precision (defined appropriately), where computational complexity grows dramatically with both precision and dimension of the state space [3], [24]–[27]. Methods to approximate discrete abstractions with approximation depth exceeding 1 have been presented for rather restricted classes of systems only, e.g. [28].

The crucial observation is the following: If the cells Ω_i are sublevel sets of maps $g_i \colon \mathbb{R}^n \to \mathbb{R}$,

$$\Omega_i = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0 \},$$

then the reachable sets $\varphi(k_0 + l, k_0 + i - 1, \Omega_i)$ are sublevel sets of maps h_i,

$$h_i(x) = g_i \left(\varphi(k_0 + l, k_0 + i - 1, \cdot)^{-1}(x) \right), \tag{7}$$

provided the inverse in (7) exists. Thus, the reachability condition (6) is fulfilled if and only if the system

$$h_1(x) \leq 0, \tag{8a}$$
$$\vdots$$
$$h_{l+1}(x) \leq 0 \tag{8b}$$

has a solution. Now, if the reachable sets $\varphi(k_0 + l, k_0 + i - 1, \Omega_i)$ can be guaranteed to be convex, (8) is often a simple convex optimization problem. Hence, as convexity of reachable sets can be guaranteed by the results of section III, discrete abstractions with arbitrary approximation depth can be readily computed up to arbitrary precision.

We would like to emphasize again that the above setting actually covers discrete-time systems with controls via the dependence on k of the right hand side of (4) and sampled versions of continuous-time systems with controls if the set of admissible controls on the sampling intervals is finite.

The remaining of this paper is structured as follows. After having introduced basic terminology in Section II, we establish convexity conditions for reachable sets of the discrete-time system (4) in Section III. The particularly important case of a discrete-time version of a continuous-time system (3) is separately investigated. It turns out that the set of states reachable at given time from a sufficiently small ellipsoid of initial states is always convex, and we provide explicit bounds on the size of those ellipsoids in terms of properties of the right hand side F of (4). In Section IV, we demonstrate the application of our convexity criteria to a discrete-time version of the pendulum equations. It turns out that the balls that lead to convex reachable sets are large enough to be used in actual computations to be performed when the two methods discussed in the present section are applied.

Due to limitation in space, the proofs of our results can only be published with an extended (journal) version of this manuscript.

II. PRELIMINARIES

$\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. $\langle\cdot|\cdot\rangle$ denotes some inner product in $\mathbb{R}^n$, $\|\cdot\|$ is the norm w.r.t. $\langle\cdot|\cdot\rangle$, and $B(x,r)$ and $\bar{B}(x,r)$ denote the open and closed, respectively, ball of radius r centered at x. ∂M denotes the boundary of $M \subseteq \mathbb{R}^n$. Two vectors x and y are perpendicular, $x \perp y$, if $\langle x|y\rangle = 0$.

The domain of a map f is denoted by $\operatorname{dom} f$, id denotes the identity map, f^{-1} is used for the inverse of f as well as for preimages. The space of linear maps $X \to Y$ is denoted by $\mathcal{L}(X,Y)$. L^* and L^T denote the adjoint and the transpose, respectively, of the linear map $L \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ with respect to $\langle\cdot|\cdot\rangle$, and $\ker L$ denotes the nullspace of L. We define $L^0 := \operatorname{id}$, and for regular L, $L^{-k} := \left(L^{-1}\right)^k$. If L is k-linear, we set $Lh^k := L(h,\dots,h)$.

Let $L \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$. By $\alpha_-(L)$ and $\alpha_+(L)$ we denote the square root of the minimum and maximum, respectively, eigenvalues of L^*L, and by $\mu_-(L)$ and $\mu_+(L)$ the minimum and maximum, respectively, eigenvalues of the self-adjoint part $\frac{1}{2}(L+L^*)$ of L.

$D^j f$ denotes the derivative of order j of f, and $D_i^j f$, the partial derivative of order j with respect to the ith argument of f, and $D_i f := D_i^1 f$, $f' := Df := D^1 f$, and $f'' := D^2 f$. C^k denotes the class of k times continuously differentiable maps.

Let $F\colon U \subseteq \mathbb{Z}\times\mathbb{R}^n \to \mathbb{R}^n$. In complete analogy to the continuous-time case, $\varphi\colon V \to \mathbb{R}^n$ is called the *general solution* of (4) if $V \subseteq \{\, (\tau,k,x) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{R}^n \mid (k,x) \in U \,\}$ and for all $(k_0,x_0) \in U$, $\varphi(\cdot,k_0,x_0)$ is the maximal solution of the initial value problem composed of (4) and the initial condition $x(k_0) = x_0$. $(k,x) \mapsto \varphi(k,0,x)$ is called the *flow* of (4) if φ is the general solution of (4) and (4) is autonomous.

We assume throughout this paper that the right hand side $F\colon U \subseteq \mathbb{Z}\times\mathbb{R}^n \to \mathbb{R}^n$ of (4) is of class C^2. Obviously then, the general solution φ of (4) is of class C^2 as well, and the map $D_3\varphi(\cdot,k_0,x)$ is a solution of the variational equation to (4), i.e., it is a matrix solution to the initial value problem

$$Z(m+1) = D_2F(m,\varphi(m,k_0,x))Z(m), \tag{9a}$$
$$Z(k_0) = \operatorname{id}. \tag{9b}$$

Consider now the linear special case

$$x(k+1) = A(k)x(k) + b(k) \tag{10}$$

of (4) and the corresponding homogeneous system

$$x(k+1) = A(k)x(k). \tag{11}$$

The *transition matrix* Φ of (11) is defined by the requirement that $\Phi(\cdot,k_0)$ is the matrix solution of the initial value problem consisting of the homogeneous system (11) and the initial condition $x(k_0) = \operatorname{id}$. Hence, $\Phi(k,k_0) = A(k-1)A(k-2)\cdots A(k_0)$ whenever $k \geq k_0$. The general solution φ of (10) is given by the formula

$$\varphi(k,k_0,x_0) = \Phi(k,k_0)x_0 + \sum_{\tau=k_0}^{k-1} \Phi(k,\tau+1)b(\tau) \tag{12}$$

for all $k,k_0 \in \mathbb{Z}$, $k \geq k_0$.

III. MAIN RESULTS

In this section, we present necessary and sufficient conditions for the convexity of a set of states reachable at some finite time from some ellipsoid of initial states through solutions of the discrete-time system (4), where $F\colon U \subseteq \mathbb{Z}\times\mathbb{R}^n \to \mathbb{R}^n$ is such that for each $k \in \mathbb{Z}$, $F(k,\cdot)$ is a C^2-diffeomorphism between two open subsets of $\mathbb{R}^n$. Throughout this section, φ denotes the general solution of (4) unless stated otherwise.

Our results are based on the following convexity criterion from [16], [18]:

III.1 Theorem. *Let $\Phi\colon U \to V$ be a C^2-diffeomorphism between open sets $U,V \subseteq \mathbb{R}^n$ and $x_0 \in U$ and $r > 0$ such that $\bar{B}(x_0,r) \subseteq U$. Then $\Phi(\bar{B}(x_0,r))$ is convex if and only if*

$$\left\langle x - x_0 \middle| \Phi'(x)^{-1}\Phi''(x)h^2 \right\rangle \leq 1 \tag{13}$$

holds for all $x \in \partial B(x_0,r)$ and all $h \perp (x-x_0)$ with $\|h\| = 1$.

Note that, in contrast to related results in [29]–[31], the conditions in Theorem III.1 are to be checked for boundary points x of $\bar{B}(x_0,r)$ and tangent vectors h only.

Our first result below gives a criterion for the reachable set $\varphi(k,k_0,\bar{B}(x_0,r))$ of (4) to be convex. Its advantage over a direct application of Theorem III.1 to the C^2-diffeomorphism $\varphi(k,k_0,\cdot)$ is that the second derivative of that diffeomorphism does not appear in the key condition (14). Hence, in order to estimate the left hand side of (14) for a particular system (4), it suffices to study the variational equation to (4).

III.2 Theorem. *Let $x_0 \in \mathbb{R}^n$, $r > 0$ and $k_0, k \in \mathbb{Z}$ and assume that $k \geq k_0$ and $\{k\} \times \{k_0\} \times \bar{B}(x_0, r) \subseteq \operatorname{dom} \varphi$. Then the reachable set $\varphi(k, k_0, \bar{B}(x_0, r))$ is convex if and only if*

$$\sum_{\tau=k_0}^{k-1} \langle x - x_0 \,|\, D_3\varphi(\tau, k_0, x)^{-1} D_2F(\tau, \varphi(\tau, k_0, x))^{-1} \cdot \\ \cdot D_2^2 F(\tau, \varphi(\tau, k_0, x))(D_3\varphi(\tau, k_0, x)h)^2 \rangle \leq 1 \quad (14)$$

for all $x \in \partial B(x_0, r)$ and all $h \perp (x - x_0)$ with $\|h\| = 1$.

The next two results are sufficient conditions for the reachable set to be convex. Their main advantage over the previous Theorem III.2 is that the bounds on the radius they establish can be determined directly from properties of the right hand side of (4).

III.3 Theorem. *Let U, F, x_0, r, k_0 and k as in Theorem III.2 and assume that the constants $d_3, \sigma_-, \sigma_+ \in \mathbb{R}$ fulfill*

$$\sigma_- \leq \alpha_- \left(D_2F(\tau, x)\right) \leq \alpha_+ \left(D_2F(\tau, x)\right) \leq \sigma_+, \quad (15)$$

$$d_3 \geq \|D_2F(\tau, x)^{-1} D_2^2 F(\tau, x)\| \quad (16)$$

for all $(\tau, x) \in U$. Then the reachable set $\varphi(k, k_0, \bar{B}(x_0, r))$ is convex if $r d_3 K(\sigma_+^2/\sigma_-) \leq 1$, where

$$K(\beta) = \begin{cases} k - k_0, & \text{if } \beta = 1, \\ \frac{\beta^{k-k_0} - 1}{\beta - 1}, & \text{otherwise}. \end{cases} \quad (17)$$

III.4 Corollary. *Let U, F, x_0, r, k_0 and k be as in Theorem III.2, K as in (17), and assume that the constants $d_2, \sigma_-, \sigma_+ \in \mathbb{R}$ fulfill (15) and*

$$d_2 \geq \|D_2^2 F(\tau, x)\|$$

for all $(\tau, x) \in U$. Then the reachable set $\varphi(k, k_0, \bar{B}(x_0, r))$ is convex if $r d_2 K(\sigma_+^2/\sigma_-) \leq \sigma_-$.

A particularly important case of (4) is that of a time-discrete version of a continuous-time system (3), where $f \colon U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. In that case, the right hand side F of (4) is given by

$$F(k, x) := \varphi((k - k_0)T + t_0, k_0 T + t_0, x) \quad (18)$$

for some $T > 0$ and $t_0 \in \mathbb{R}$, where φ is the general solution of (3) and $\{(k - k_0)T + t_0\} \times \{k_0 T + t_0\} \times \{x\} \subseteq \operatorname{dom} \varphi$. The requirement that $F(k, \cdot)$ be a diffeomorphism of class C^2 is then fulfilled whenever U is open and the right hand side f of (3) is of class C^2. Moreover, it follows from [18] that the conditions in Theorem III.3 can be easily verified by inspection of the right hand side f of (3):

III.5 Proposition. *Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be open and the right hand side f of (3) be of class C^2. Let further $x_0 \in \mathbb{R}^n$, $r > 0$ and $t_0, t \in \mathbb{R}$ be such that $t \geq t_0$ and $\{t\} \times \{t_0\} \times \bar{B}(x_0, r) \subseteq \operatorname{dom} \varphi$, where φ is the general solution of (3). Finally, assume that the constants $c_2, \lambda_-, \lambda_+ \in \mathbb{R}$ fulfill*

$$\lambda_- \leq \mu_- \left(D_2 f(\tau, x)\right) \leq \mu_+ \left(D_2 f(\tau, x)\right) \leq \lambda_+, \quad (19)$$

$$c_2 \geq \|D_2^2 f(\tau, x)\| \quad (20)$$

for all $(\tau, x) \in U$. Then

$$\|D_3\varphi(t, t_0, x)\| \leq \mathrm{e}^{\lambda_+(t - t_0)}, \quad (21)$$

$$\|D_3\varphi(t, t_0, x)^{-1}\| \leq \mathrm{e}^{-\lambda_-(t - t_0)}, \quad (22)$$

$$\|D_3\varphi(t, t_0, x)^{-1} D_3^2 \varphi(t, t_0, x)\| \leq c_2 \widetilde{K}(2\lambda_+ - \lambda_-),$$

for all $x \in \bar{B}(x_0, r)$, where

$$\widetilde{K}(\beta) = \begin{cases} t - t_0, & \text{if } \beta = 0, \\ \left(\exp(\beta(t - t_0)) - 1\right)/\beta, & \text{otherwise}. \end{cases}$$

III.6 Corollary. *Let U, f, x_0, r, t_0, t, c_2, $\widetilde{K}$, λ_- and λ_+ as in Proposition III.5, let $T = t - t_0$, $k \geq k_0$ and the right hand side F of (4) be given by (18), and let φ be the general solution of (4).*
Then the reachable set $\varphi(k, k_0, \bar{B}(x_0, r))$ is convex if $r c_2 \widetilde{K}((k - k_0)(2\lambda_+ - \lambda_-)) \leq 1$.

We would like to comment on how to verify the hypotheses in the preceding results if the ball $\bar{B}(x_0, r)$ of initial values is an ellipsoid rather than an Euclidean ball, or equivalently, if the inner product $\langle \cdot | \cdot \rangle$ is different from the Euclidean inner product $(\cdot|\cdot)$ given by

$$(x|y) = \sum\nolimits_{i=1}^{n} x_i y_i. \quad (23)$$

Obviously, condition (15) reduces to a bound on the eigenvalues of $D_2F(\tau, x)^* D_2F(\tau, x)$, and condition (19), to a bound on the eigenvalues of the self-adjoint part $\frac{1}{2}(D_2 f(\tau, x) + D_2 f(\tau, x)^*)$ of $D_2 f(\tau, x)$. If $\langle \cdot | \cdot \rangle$ is Euclidean, $L^* = L^T$ for all $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $\alpha_\pm(L)$ are the minimum and maximum, respectively, singular values of L. If $\langle \cdot | \cdot \rangle$ is not Euclidean, there is a symmetric positive definite matrix Q such that $\langle x | y \rangle = (x | Qy)$, from which $L^* = Q^{-1} L^T Q$ follows for all $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, so that conditions (15) and (19) can be readily verified.

IV. EXAMPLE

In this section, we demonstrate the application of our results to a discrete-time version of the pendulum equations. To this end, we define the right hand side F of (4) by

$$F(k, x) := \psi(k/4, x), \quad (24)$$

where $\psi \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is the flow of the pendulum equations

$$\dot{x}_1 = x_2, \quad (25a)$$

$$\dot{x}_2 = -\omega^2 \sin(x_1) - 2\gamma x_2. \quad (25b)$$

The investigation of the convexity of reachable sets of more general systems, such as a cart-pole system with piecewise constant control, can be reduced to the autonomous system (25) [16].

For the sake of simplicity, we restrict ourselves to the case of the Euclidean inner product in $\mathbb{R}^n$ defined in (23) in this section.

We first demonstrate the application of Cor. III.6:

TABLE I

BOUNDS ON THE RADIUS OF THE BALL Ω THAT ENSURES CONVEXITY OF THE REACHABLE SET $\varphi(k, \Omega)$ OVER k. (ROUNDED TO TWO DECIMAL PLACES.)

k	1	2	3	4
R (Cor. IV.1)	2.7	.86	.35	.16
R (Th. IV.2)	3.4	1.1	.45	.20
R (numerical)	3.9	1.8	.98	.52

IV.1 Corollary. *Let φ be the flow of (4), the right hand side F of (4) be given by (24), and let $k > 0$ and d_1 and R be given by*

$$d_1 = -\gamma + 3\sqrt{\gamma^2 + (1+\omega^2)^2/4}, \tag{26}$$

$$R = \frac{d_1}{\omega^2(\exp(kd_1/4) - 1)}. \tag{27}$$

Then the image of any ball with radius not exceeding R under the map $\varphi(k, \cdot)$ is convex.

The next result, which we obtain by applying Theorem III.3 and Proposition III.5, improves the bound (27) of Corollary IV.1. Theorem IV.2 also improves a result in [18] as it yields a valid bound on the radius for every $k > 0$. In contrast, with the tools developed in [18] we had obtained a related result on a finite time interval only.

IV.2 Theorem. *Let φ be the flow of (4), the right hand side F of (4) be given by (24), let $k > 0$, d_1 be given by (26), set $\rho = \sqrt{\omega^2 + \gamma^2}$ and*

$$R = \frac{6\omega\rho(\mathrm{e}^{d_1/4} - 1)}{(1 + (\omega+\gamma)^2)^{3/2}\,(\mathrm{e}^{kd_1/4} - 1)} \cdot \frac{1}{\sinh(\rho/4)(\cosh(\rho/2) + 5 - 10\exp(-\omega))}, \tag{28}$$

and assume $0 \leq \gamma \leq \omega$ and $1 \leq \omega \leq \pi$. Then the image of any ball with radius not exceeding R under the map $\varphi(k, \cdot)$ is convex.

Tab. I shows the bounds obtained from Corollary IV.1 and Theorem IV.2 in comparison to a bound obtained numerically for the undamped mathematical pendulum ($\omega = 1$, $\gamma = 0$). The figures show that the balls that lead to convex reachable sets are large enough to be used in actual computations to be performed when the two methods discussed in Section I are applied.

We would like to emphasize that the results from this section are of a global type, i.e., convexity of reachable sets $\varphi(k, \bar{B}(x_0, r))$ for arbitrary $x_0 \in \mathbb{R}^2$ is guaranteed, provided that $r \leq R$.

V. CONCLUSIONS

We have obtained necessary and sufficient conditions for reachable sets of discrete-time systems (4) to be convex. Our results imply that outer discrete approximations with arbitrary approximation depth can be readily computed up to arbitrary precision. A further potential application of our results is outer polyhedral approximation of reachable sets, which becomes almost universally applicable if the reachable set is known to be convex.

Extension of our results to infinite dimensions, to the case of $C^{1,1}$ smoothness (C^1 with Lipschitz-continuous derivative), and to arbitrary sublevel sets of initial states seems to be possible with the help of the results of [17], [18]. An open question is how to relax our standing assumption that $F(k, \cdot)$ be a diffeomorphism. In addition, the two applications of our results presented in Section I require further investigation, e.g. regarding approximation error, the possibility to obtain non-polyhedral approximations, and computational efficiency.

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