Computation of discrete abstractions of arbitrary memory span for nonlinear sampled systems

Gunther Reißig*

Technische Universität Berlin, Fakultät Elektrotechnik und Informatik,
Heinrich-Hertz-Lehrstuhl für Mobilkommunikation HFT 6, Einsteinufer 25, D-10587
Berlin, Germany, http://www.reiszig.de/gunther/

Abstract. In this paper, we present a new method for computing discrete abstractions of arbitrary memory span for nonlinear sampled systems with quantized output. In our method, abstractions are represented by collections of conservative approximations of reachable sets by polyhedra, which in turn are represented by collections of half-spaces. Important features of our approach are that half-spaces are shared among polyhedra, and that the determination of each half-space requires the solution of a single initial value problem in an ordinary differential equation over a single sampling interval only. Apart from these numerical integrations, the only nontrivial operation to be performed repeatedly is to decide whether a given polyhedron is empty. In particular, in contrast to previous approaches, there are no intermediate bloating steps, and convex hulls are never computed. Our method heavily relies on convexity of reachable sets and applies to any sufficiently smooth system if either the sampling period, or the system of level sets of the quantizer can be chosen freely. In particular, it is not required that the system to be abstracted have any stability properties.

Key words: discrete abstractions, polyhedral approximations, reachable sets

1 Introduction

A well-known method for the solution of analysis and synthesis problems for continuous, discrete and hybrid systems consists in first computing a *discrete abstraction* of the system's behavior in the sense of Willems [1,2], and then solving a corresponding (auxiliary) problem for the abstraction, e.g. [3,4,5,6,7,8]. Here, the term *abstraction* refers to a conservative approximation, i.e., a superset, of the system's behavior, which is called *discrete* if it can be realized by a finite

* This a preprint of a paper published in R. Majumdar and P. Tabuada (Eds.): HSCC 2009, LNCS 5469, pp. 306-320, Springer-Verlag, 2009. You may find the most current version of this file together with a BibTEX entry at author's homepage. The definitive publication is available at http://www.springerlink.com.

(in general non-deterministic) automaton. Auxiliary problems arising in this way are solvable controller synthesis problems if the original problem is and the abstraction is sufficiently accurate. Solvability may be verified, and solutions may be obtained using well-known algorithms from discrete mathematics [9,10,11,8]. In addition, under mild assumptions, it follows from the conservativeness of the approximation that any solution of the auxiliary problem will also be a solution to the problem for the original system, e.g. [5,6,7,8]

One of the most complex steps in the above approach is the computation of sufficiently accurate discrete abstractions, which is equivalent to conservative approximation of a large number of reachable sets [6]. Known methods are restricted to rather limited classes of systems or to abstractions of memory span 1, lead to overly conservative abstractions, suffer from their prohibitive computational complexity, or require the solution of non-convex optimization or optimal control problems, see [12,13,14,15,16,17,18,19,20,21] and the references given there. In this paper, we present a new method for computing discrete abstractions of arbitrary memory span for nonlinear sampled systems with quantized output.

In our method, abstractions are represented by collections of conservative approximations of reachable sets by polyhedra, which in turn are represented by collections of half-spaces: We start from a collection of conservative polyhedral approximations of the level sets of the quantizer, which represents a discrete abstraction of memory span 0, and then iteratively determine conservative polyhedral approximations of the reachable sets that define abstractions of greater memory span. Important features of our approach are that half-spaces are shared among polyhedra, and that the determination of each half-space requires the solution of a single initial value problem in an ordinary differential equation over a single sampling interval only. Apart from these numerical integrations, the only nontrivial operation to be performed repeatedly is to decide whether a given polyhedron is empty. In particular, in contrast to previous approaches, there are no intermediate bloating steps, and convex hulls are never computed. Our method heavily relies on convexity of reachable sets and applies to any sufficiently smooth system if either the sampling period, or the system of level sets of the quantizer can be chosen freely. In particular, it is not required that the system to be abstracted have any stability properties.

The remaining of this paper is structured as follows: In section 2 we define the class of sampled systems with quantized output, for which we shall develop an efficient algorithm for computing discrete abstractions. In section 3 we characterize the smallest of those abstractions in terms of reachable sets. In section 4, we present an efficient algorithm for computing discrete abstractions for the class of systems introduced in section 2, under the assumption that all relevant reachable sets are convex. We also discuss two recent results from [22,23] from which convexity of reachable sets can be deduced under mild conditions. Finally, we apply our method to the problem of swinging up the mathematical pendulum in section 5. Proofs of our results will be published with an extended (journal) version of this manuscript.

2 Sampled systems with quantized output

Let the control system

$$\dot{x} = f(x, u(t)) \tag{1}$$

with $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, a sampling period $T > 0$, and a finite set $U \subseteq (\mathbb{R}^m)^{[0,T]}$ of admissible controls on sampling intervals be given. Hence, elements of U are signals $[0, T] \to \mathbb{R}^m$, and we identify each sequence $(u_k)_{k\in\mathbb{Z}_+}$ of such signals with a control signal defined on $\mathbb{R}_+$,

$$(u_0, u_1, \dots)(t) := u_{\lfloor t/T \rfloor}(t - \lfloor t/T \rfloor) \quad \text{for all } t \geq 0,$$

see Fig. 1. Here, $\lfloor x \rfloor$ denotes the largest integer not greater than x, $\mathbb{R}_+$ and $\mathbb{Z}_+$, the set of nonnegative reals and integers, respectively, and A^B, the set of maps $B \to A$.

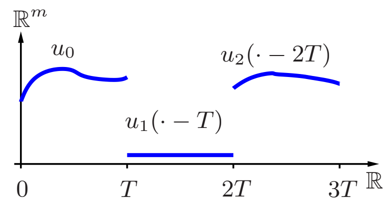

Fig. 1. An admissible control signal for (1).

The set $\mathcal{U}$ of controls $u\colon \mathbb{R}_+ \to \mathbb{R}^m$ admissible for (1) is now defined as the set of all sequences in U,

$$\mathcal{U} = \left\{ u\colon \mathbb{R}_+ \to \mathbb{R}^m \,\middle|\, \forall_{k\in\mathbb{Z}_+} \exists_{u_k\in U} \forall_{t\in[kT,(k+1)T[} \; u(t) = u_k(t - kT) \right\}. \tag{2}$$

Here, $[a, b]$, $]a, b[$, and $[a, b[$, $]a, b]$ denote closed, open and half-open intervals, respectively.

We assume throughout this paper that for any admissible control $u \in \mathcal{U}$, initial value problems composed of (1) and an initial condition

$$x(0) = x_0 \tag{3}$$

are uniquely solvable for any $x_0 \in \mathbb{R}^n$, with all solutions extendable to the entire time axis $\mathbb{R}_+$.

As an extension of the well-known concepts of flow and general solution for ordinary differential equations [24,25], we define the *general solution* φ of (1) by

$$\varphi(t, x_0, u) := \text{value of the solution of initial value problem (1), (3) at time } t,$$

where $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}$. Note that it is not necessary to specify all the components u_k of $u = (u_0, u_1, \dots)$, and that the values of u at sampling instants are irrelevant, so we may write

$$\varphi(t, x_0, u_0, \dots, u_k) := \varphi(t, x_0, u) \quad \text{if } t \leq (k + 1)T,\ u = (u_0, \dots, u_k, \dots).$$

We now consider the sampled version

$$x(k+1) = \varphi(T, x(k), u_k), \quad k \in \mathbb{Z}_+ \tag{4}$$

of (1), where φ is the general solution of (1). By our assumptions on (1), the right hand side of the difference equation (4) is defined for all $(x(k), u_k) \in \mathbb{R}^n \times U$.

We define the quantizer by specifying its level sets: Let $C \subseteq \mathcal{P}(\mathbb{R}^n)$ be a finite covering of the state space $\mathbb{R}^n$ of (4) that does not contain the empty set, where $\mathcal{P}(M)$ denotes the power set of M. The quantizer Q is defined as the map

$$Q \colon \mathbb{R}^n \to \mathcal{P}(C) \colon x \mapsto \{\, \Omega \in C \,|\, x \in \Omega \,\}. \tag{5}$$

Note that $Q(x) \neq \emptyset$ for any $x \in \mathbb{R}^n$ as C is a covering of $\mathbb{R}^n$, and that, in general, the quantizer is non-deterministic. See [13] for the equivalent concept of a "measurement map".

The system composed of the sampled system (4) and the quantizer (5) shown

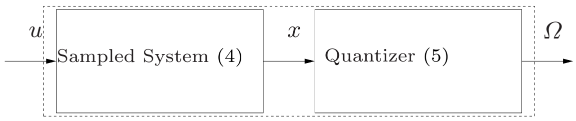

Fig. 2. Sampled system with quantized output.

in Fig. 2 may be described by the following difference equation with set valued output:

$$x(k+1) = \varphi(T, x(k), u_k), \quad k \in \mathbb{Z}_+, \tag{6a}$$

$$\Omega_k \in Q(x(k)). \tag{6b}$$

The *(external) behavior* [1,2] $B_{(6)}$ of the system (6) is the set of all (external) signals $(u, \Omega) \in U^{\mathbb{Z}_+} \times C^{\mathbb{Z}_+}$ that are compatible with (6), i.e.,

$$B_{(6)} = \left\{\, (u, \Omega) \,\middle|\, \exists_{x \colon \mathbb{Z}_+ \to \mathbb{R}^n} \, \forall_{k \in \mathbb{Z}_+} \; (x(k) \in \Omega_k \text{ and } x(k+1) = \varphi(T, x(k), u_k)) \,\right\}. \tag{7}$$

3 Smallest discrete abstractions and reachable sets

In this section we characterize the smallest N-complete discrete abstraction of the behavior $B_{(6)}$ given by (7) in terms of reachable sets of the time-continuous control system (1). We begin with some terminology from behavioral theory [2]:

Let a set $I \subseteq \mathbb{Z}_+$, an arbitrary set X and a behavior $B \subseteq X^{\mathbb{Z}_+}$ be given. The *restriction of* B *to* I, $B|_I$, is defined by $B|_I := \{\, x|_I \,|\, x \in B \,\}$, where $x|_I$ denotes the restriction of the map x to I.

B is called *time-invariant* if $\sigma B \subseteq B$, where σ denotes the shift operator defined by $\sigma := \sigma^1$ and $(\sigma^k x)(t) := x(t+k)$ for all $x\colon \mathbb{Z}_+ \to X$ and all $k, t \in \mathbb{Z}_+$.

If B is time-invariant, it is called *complete* if

$$B = \left\{ x \in X^{\mathbb{Z}_+} \,\middle|\, \forall_{k_1,k_2 \in \mathbb{Z}_+, k_1 \leq k_2}\ x|_{[k_1,k_2]} \in B|_{[k_1,k_2]} \right\},$$

and it is called *complete with memory span* N (or N*-complete*, for short) for some $N \in \mathbb{Z}_+$ if

$$B = \left\{ x \in X^{\mathbb{Z}_+} \,\middle|\, \forall_{k \in \mathbb{Z}_+}\ (\sigma^k x)|_{[0,N]} \in B|_{[0,N]} \right\}.$$

If B is time-invariant, we call the set

$$\bigcap_{\substack{B \subseteq B' \subseteq X^{\mathbb{Z}_+}, \\ B' \text{ is } N\text{-complete}}} B' \tag{8}$$

the N*-complete hull* of B. (The map that assigns to B its N-complete hull (8) is a closure operator [26], which is why we call (8) a *hull*; N-complete hulls are called "strongest N-complete approximations" in [6].)

The behavior $B_{(6)}$ of the sampled system with quantized output is time-invariant, but in general not complete, which is why we are looking for a discrete abstraction of it. For any $N \in \mathbb{Z}_+$, the N-complete hull of $B_{(6)}$ is the smallest abstraction of the kind we seek to obtain. Unfortunately, that abstraction may be computed exactly for special classes of systems (1) and quantizers (5) only. Nevertheless, the following characterizations of that smallest abstraction will be useful in the next section when we derive an algorithm for effectively computing another abstraction that conservatively approximates the smallest one.

Theorem 1. *Let $N \in \mathbb{Z}_+$ and B_N be the N-complete hull of the behavior $B_{(6)}$ given by (7), φ the general solution of (1), and $(u, \Omega) \in U^{\mathbb{Z}_+} \times C^{\mathbb{Z}_+}$. Then the following are equivalent:*

(i) $(u, \Omega) \in B_N$.

(ii) For all $\tau \in \mathbb{Z}_+$ there exists $x_0 \in \mathbb{R}^n$ such that $\varphi(kT, x_0, u_\tau, \ldots, u_{\tau+k-1}) \in \Omega_{\tau+k}$ holds for all $k \in \{0, \ldots, N\}$.

(iii) For all $\tau \in \mathbb{Z}_+$ the following holds:

$$\Omega_{\tau+N} \cap \bigcap_{k=1}^{N} \varphi(kT, \Omega_{\tau+N-k}, u_{\tau+N-k}, \ldots, u_{\tau+N-1}) \neq \emptyset. \tag{9}$$

(iv) $M_N^\tau \neq \emptyset$ for all $\tau \in \mathbb{Z}_+$, where M_N^τ is defined by

$$M_0^\tau = \Omega_\tau,$$
$$M_k^\tau = \Omega_{\tau+k} \cap \varphi(T, M_{k-1}^\tau, u_{\tau+k-1}) \quad (k \in \{1, \ldots, N\}).$$

Characterization (iv) has been given in [6], and a characterization similar to (iii) has been proposed in [23].

A set of the form $\varphi(t, \Omega, u)$ arising in Theorem 1 is called *reachable set from Ω at time t under control u.*

4 Computation of discrete abstractions and convexity of reachable sets

We have seen in Section 3 that computing the smallest discrete abstraction of a particular memory span for the behavior $B_{(6)}$ given by (7) requires the solution of numerous difficult reachability problems that may be solved exactly for special classes of systems (1) and quantizers (5) only. In the present section, we aim at computing another discrete abstraction which approximates the smallest one. Our starting point is the following basic idea:

If $\Omega_{\tau+N}$ and all the reachable sets on the left hand side of condition (9) in Theorem 1 were convex, these sets could be substituted with approximations by means of supporting half-spaces. The resulting approximate condition would then characterize a superset of the N-complete hull of $B_{(6)}$ that is N-complete, and hence, a discrete abstraction of memory span N of $B_{(6)}$.

In the remaining of this section, we shall derive an algorithm for the computation of discrete abstractions of arbitrary memory span N for the behavior $B_{(6)}$ given by (7) which approximates the N-complete hull of $B_{(6)}$. To this end, we start with the question of how to obtain conservative polyhedral approximations of reachable sets by means of supporting half-spaces.

Definition 1. *Let $\Omega \subseteq \mathbb{R}^n$ be convex and $p \in \Omega$. A vector $v \in \mathbb{R}^n$ is* normal to Ω at p *[27] if $\langle v|x-p\rangle \leq 0$ for all $x \in \Omega$, where $\langle \cdot|\cdot \rangle$ denotes the standard Euclidean inner product.*

Proposition 1. *Let the right hand side f of (1) be of class C^1 w.r.t. its first argument and continuous, and let φ denote the general solution of (1). Let further $u \in \mathcal{U}$ be a piecewise continuous control admissible for (1), $\Omega \subseteq \mathbb{R}^n$ be convex, $p \in \Omega$, $v \in \mathbb{R}^n$, and $\tau \in \mathbb{R}_+$. Finally, let v' be the value at time τ of the solution of the following initial value problem:*

$$\begin{aligned} \dot{x} &= -D_1 f(\varphi(t,p,u), u(t))^* x, \\ x(0) &= v, \end{aligned}$$

where $(\cdot)^*$ denotes the transpose, and D_1, the partial derivative w.r.t. the first argument.
If the reachable set $\varphi(\tau,\Omega,u)$ is convex, then v is normal to Ω at p if and only if v' is normal to $\varphi(\tau,\Omega,u)$ at $\varphi(\tau,p,u)$.

The above result tells us that conservative polyhedral approximations of all the reachable sets on the left hand side of condition (9) may be obtained from analogous approximations of the level sets of the quantizer (5) by solving initial value problems in the $2n$-dimensional ordinary differential equation

$$\dot{x} = f(x, u(t)), \tag{10a}$$
$$\dot{y} = -D_1 f(x, u(t))^* y. \tag{10b}$$

For further reference, we define a map φ^* that realizes the determination of a supporting half-space of the reachable set $\varphi((k+1)T, \Omega, u)$ from a supporting half-space of Ω ($k \in \mathbb{Z}_+$):

$$\varphi^* \colon \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^n \colon (p, v, u_0, \ldots, u_k) \mapsto \psi((k+1)T, (p, v), u_0, \ldots, u_k), \tag{11}$$

where ψ is the general solution of (10) and T, the sampling period.

We now formalize the substitution of reachable sets on the left hand side of condition (9) in Theorem 1 with approximations by means of supporting half-spaces:

Definition 2. *Let P be the map defined by*

$$P \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n) \colon (p, v) \mapsto \{\, x \in \mathbb{R}^n \mid \langle v | x - p \rangle \leq 0 \,\}$$

and set

$$P(\Sigma) = \bigcap_{(p,v) \in \Sigma} P(p, v)$$

for $\Sigma \subseteq \mathbb{R}^n \times \mathbb{R}^n$.
Let $\Sigma \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$.
We call Σ a conservative polyhedral approximation of Ω if $\Omega \subseteq P(\Sigma)$, and a supporting polyhedral approximation of Ω, if $p \in \Omega$ and v is normal to Ω at p, for all $(p, v) \in \Sigma$.
If Σ is a conservative polyhedral approximation of Ω, $(p, v) \in \Sigma$ is called redundant in Σ if $P(\Sigma) = P(\Sigma \setminus \{(p, v)\})$.

Let $N \in \mathbb{Z}_+$, $\Omega_0, \ldots, \Omega_N \subseteq \mathbb{R}^n$, and $u_0, \ldots, u_{N-1} \in U$ and define

$$M(\Omega_0, \ldots, \Omega_N, u_0, \ldots, u_{N-1}) := \Omega_N \cap \bigcap_{k=1}^{N} \varphi(kT, \Omega_{N-k}, u_{N-k}, \ldots, u_{N-1}).$$

Hence, $M(\Omega_\tau, \ldots, \Omega_{\tau+N}, u_\tau, \ldots, u_{\tau+N-1})$ is just the set on the left hand side of condition (9). For any convex $\Omega \subseteq \mathbb{R}^n$, let $\Sigma(\Omega)$ be a supporting polyhedral approximation of Ω and define

$$\widehat{M}(\Omega_0, \ldots, \Omega_N, u_0, \ldots, u_{N-1}) := P(\Sigma(\Omega_N)) \cap \bigcap_{k=1}^{N} P\left(\varphi^*(\Sigma(\Omega_{N-k}), u_{N-k}, \ldots, u_{N-1})\right).$$

Hence, $\widehat{M}(\Omega_\tau, \ldots, \Omega_{\tau+N}, u_\tau, \ldots, u_{\tau+N-1})$ is the left hand side of condition (9) with all the reachable sets substituted with supporting polyhedral approximations obtained from application of the map φ^* to supporting polyhedral approximations of level sets of the quantizer.

We now define

$$S(\Omega_0) := \Sigma(\Omega_0), \tag{12}$$

$$S(\Omega_0, \dots, \Omega_k, u_0, \dots, u_{k-1}) := \Sigma(\Omega_k) \cup \bigcup_{q=1}^{k} \varphi^*(\Sigma(\Omega_{k-q}), u_{k-q}, \dots, u_{k-1}) \tag{13}$$

for all $k \in \{1, \dots, N\}$ to obtain

$$\widehat{M}(\Omega_0, \dots, \Omega_N, u_0, \dots, u_{N-1}) = P(S(\Omega_0, \dots, \Omega_N, u_0, \dots, u_{N-1})).$$

Hence, $S(\Omega_0, \dots, \Omega_N, u_0, \dots, u_{N-1})$ is a conservative polyhedral approximation of $\widehat{M}(\Omega_0, \dots, \Omega_N, u_0, \dots, u_{N-1})$, though not necessarily a supporting one.

The following result shows that the sets $S(\dots)$ just defined enjoy a recursive description analogous to the one for the sets M_k^τ. (See condition (iv) in Theorem 1.)

Theorem 2. *Let $N \in \mathbb{Z}_+$, $\Omega_0, \dots, \Omega_N \subseteq \mathbb{R}^n$ be convex, and $u_0, \dots, u_{N-1} \in U$. Let φ be the general solution of (1) and assume that the reachable sets $\varphi(kT, \Omega_{N-k}, u_{N-k}, \dots, u_{N-1})$ are convex for all $k \in \{1, \dots, N\}$. For each $k \in \{0, \dots, N\}$, let $\Sigma(\Omega_k)$ be a supporting polyhedral approximation of Ω_k, and let $S(\Omega_0)$, ..., $S(\Omega_0, \dots, \Omega_N, u_0, \dots, u_{N-1})$ be defined by (12)-(13). Then*

$$S(\Omega_0, \dots, \Omega_k, u_0, \dots, u_{k-1}) = \Sigma(\Omega_k) \cup \varphi^*(S(\Omega_0, \dots, \Omega_{k-1}, u_0, \dots, u_{k-2}), u_{k-1})$$

for all $k \in \{1, \dots, N\}$.
Furthermore, if (p, v) is redundant in $S(\Omega_0, \dots, \Omega_{k-1}, u_0, \dots, u_{k-2})$, then it is so in $S(\Omega_0, \dots, \Omega_k, u_0, \dots, u_{k-1})$.

It follows from the above result that half-spaces are shared among many of the conservative polyhedral approximations $\widehat{M}(\dots)$ of reachable sets and that the computational cost per half-space is just a single solution of an initial value problem in the $2n$-dimensional ordinary differential equation (10) over a single sampling interval.

We now propose an algorithm that, under the assumption that reachable sets are convex, determines discrete abstractions of the behavior $B_{(6)}$ given by (7).
Input:

(i) $N \in \mathbb{Z}_+$ (memory span of abstraction to be computed);
(ii) $T, \mathcal{U}, U$ (see section 2);
(iii) C: finite covering of $\mathbb{R}^n$ by convex polyhedra (level sets of the quantizer (5));
(iv) $C' := \{\, \Omega \in C \mid \Omega \text{ bounded} \,\}$;
(v) a set $\widehat{\Omega}$ for each $\Omega \in C'$ with $\Omega \subseteq \widehat{\Omega}$ and reachable sets $\varphi(kT, \widehat{\Omega}, u)$ convex for all $k \in \{0, \dots, N\}$, $\Omega \in C'$ and $u \in \mathcal{U}$;
(vi) a supporting polyhedral approximation $\Sigma(\widehat{\Omega})$ of $\widehat{\Omega}$ for all $\Omega \in C'$.

```
1: for all Ω ∈ C' do
2:    S̃(Ω) = Σ(Ω̂)
```

3: **end for**
4: **for all** $\Omega \in C \setminus C'$ **do**
5: $\quad \widetilde{S}(\Omega) = \emptyset$
6: **end for**
7: **for** $k = 1, \ldots, N$ **do**
8: $\quad$ **for all** $(\Omega_0, \ldots, \Omega_k, u_0, \ldots, u_{k-1}) \in C^{k+1} \times U^k$ **do**
9: $\quad\quad$ **if** $\widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2}) = \emptyset$ **then**
10: $\quad\quad\quad Z := \emptyset$
11: $\quad\quad$ **else if** $\Omega_k \cap P(\varphi^*(\widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2}), u_{k-1})) = \emptyset$ **then**
12: $\quad\quad\quad Z := \mathbb{R}^n \times \mathbb{R}^n$
13: $\quad\quad$ **else if** $\Omega_k \notin C'$ **then**
14: $\quad\quad\quad Z := \emptyset$
15: $\quad\quad$ **else**
16: $\quad\quad\quad Z := \Sigma(\widehat{\Omega}_k) \cup \varphi^*(\widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2}), u_{k-1})$
17: $\quad\quad$ **end if**
18: $\quad\quad \widetilde{S}(\Omega_0, \ldots, \Omega_k, u_0, \ldots, u_{k-1}) := Z$
19: $\quad$ **end for**
20: **end for**
Output: $\widetilde{S}(\ldots)$.

The following result shows that the above algorithm determines a discrete abstraction of the behavior $B_{(6)}$ of the sampled and quantized system (6).

Theorem 3. *Denote by $\widetilde{S}(\ldots)$ the sets determined by the above algorithm. Under the assumptions made in the list of inputs, the set*

$$\left\{ (u, \Omega) \in U^{\mathbb{Z}_+} \times C^{\mathbb{Z}_+} \,\middle|\, \forall_{\tau \in \mathbb{Z}_+} \; P(\widetilde{S}(\Omega_\tau, \ldots, \Omega_{\tau+N}, u_\tau, \ldots, u_{\tau+N-1})) \neq \emptyset \right\}$$

is an N-complete conservative approximation of the behavior $B_{(6)}$ given by (7).

Some remarks are in order. First, note that the algorithm contains just two nontrivial operations which need to be performed repeatedly, namely, the determination of the set

$$\varphi^*(\widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2}), u_{k-1}), \tag{14}$$

which appears at lines 11 and 16, and the test for emptiness at line 11. Regarding the former operation, it follows from the definition (11) of the map φ^* that for each $s \in \widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2})$, determination of (14) requires the solution of an initial value problem in the $2n$-dimensional ordinary differential equation (10) over a single sampling interval. Hence, the set (14) may be determined from at most $|\widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2})|$ such solutions, where $|\cdot|$ denotes cardinality. As $P(\varphi^*(\widetilde{S}(\Omega_0, \ldots, \Omega_{k-1}, u_0, \ldots, u_{k-2}), u_{k-1}))$ is a convex polyhedron, the test for emptiness at line 11 may also be effectively performed since Ω_k is also a convex polyhedron by hypothesis (iii) in the list of inputs of the algorithm.

Second, it should be obvious that the sets $\emptyset$ and $\mathbb{R}^n \times \mathbb{R}^n$ play a role similar to zeros in sparse matrices [28,29]. In particular, if the sets $\widetilde{S}(\dots)$ are stored in a tree, sets $\widetilde{S}(\dots) = \emptyset$ and $\widetilde{S}(\dots) = \mathbb{R}^n \times \mathbb{R}^n$ do not need to be stored and computations on them do not need to actually be performed.

Finally, note that all our arguments so far were based on the *assumption* that reachable sets arising in characterizations of N-complete hulls are convex. It follows from recent results of the author [22,23] that convexity of reachable sets can be guaranteed under mild smoothness assumptions on the right hand side f of the continuous control system (1). Let us briefly look at special cases of two such results from [22].

Theorem 4. *Let the right hand side f of (1) be of class $C^{1,1}$ (C^1 with Lipschitz derivative) with respect to its first argument and continuous, and let $u \in \mathcal{U}$ be piecewise continuous. Let further $x_0 \in \mathbb{R}^n$, $r > 0$ and $t \geq 0$. Finally, assume that*

$$M_1 \geq 2\mu_+ \left(D_1 f(x, u(\tau))\right) - \mu_- \left(D_1 f(x, u(\tau))\right) \tag{15}$$

holds for all $(\tau, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, and let M_2 be a Lipschitz constant for the map $(\tau, x) \mapsto D_1 f(x, u(\tau))$ w.r.t. its second argument on $\mathbb{R}_+ \times \mathbb{R}^n$. Then the reachable set $\varphi(t, \bar{B}(x_0, r), u)$ is convex if

$$r M_2 \int_0^t \mathrm{e}^{M_1 \tau} d\tau \leq 1. \tag{16}$$

Here, $\mu_+(M)$ and $\mu_-(M)$ denote the maximum and minimum, respectively, eigenvalues of the symmetric part $(M + M^*)/2$ of M, and $\bar{B}(x_0, r)$, the closed Euclidean ball of radius r centered at x_0 w.r.t. the Euclidean norm $\|\cdot\|$.

Theorem 5. *Let u, f, x_0, r, and t as in Theorem 4 and assume in addition that f is of class C^2 with respect to its first argument. Then $\varphi(t, \bar{B}(x_0, r), u)$ is convex if and only if*

$$\int_0^t \left\langle x - x_0 \middle| D_2\varphi(\tau, x, u)^{-1} D_1^2 f(\varphi(\tau, x, u), u(\tau))(D_2\varphi(\tau, x, u)h)^2 \right\rangle d\tau \leq 1 \tag{17}$$

for all $x \in \partial \bar{B}(x_0, r)$ and all $h \perp (x - x_0)$ with $\|h\| = 1$. Here, ∂X denotes the boundary of X, D_1^2, the second order partial derivative w.r.t. the first argument, and $D_1^2 f(x, u)h^2 := D_1^2 f(x, u)(h, h)$.

The bounds M_1 and M_2 in Theorem 4 may be directly determined from the right hand side f of the time-continuous system (1) and the set $\mathcal{U}$ of admissible controls, and the bound (16) on the radius is sharp provided $n \geq 2$ [22]. Application of previous results from [30,31,32] would necessarily be based on estimates of $\|D_2\varphi(t, \cdot, u)^{-1}\|$ and $\|D_2^2\varphi(t, \cdot, u)\|$ and, in general, would yield a smaller bound.

Theorem 4 implies that the reachable set $\varphi(t, \bar{B}(x_0, r), u)$ is convex whenever t or r is sufficiently small. Hence, if either the sampling period T, or the system of level sets of the quantizer (5) can be chosen freely, convexity of reachable sets arising in the algorithm proposed in this section, and hence, applicability of the algorithm, can be guaranteed by either choosing T sufficiently small or choosing sufficiently small balls as the elements of C'.

While Theorem 4 gives a sufficient condition for the convexity of a reachable set, Theorem 5 appears to have the form of a criterion. However, condition (17) contains the general solution φ of (1) and therefore, may only rarely be directly verified. Instead, one usually has to resort to estimating the integrand on the left hand side of (17). In the twice continuously differentiable case, use of the estimate obtained from Ważewski's inequality [22] would yield precisely the bound (16) in Theorem 4. The advantage of Theorem 5 is that for specific examples of (1) one is often able to obtain better estimates for the integrand in (17), and hence, larger bounds on the radius than (16). This has been demonstrated in [22]. In view of the algorithm proposed in this section, note that larger bounds directly translate into lower computational complexity.

As Theorems 4 and 5 extend to right hand sides f defined in arbitrary Hilbert spaces, convexity of reachable sets from ellipsoids rather than from Euclidean balls may also be guaranteed [22]. In view of these results, each bounded element $\Omega \in C$ of the system C of level sets of the quantizer (5) should be contained in some ellipsoid $\widehat{\Omega}$ whose reachable sets are guaranteed to be convex by Theorems 5 and 4. See items (iv) and (v) of the list of inputs of the algorithm.

5 Example

Consider the pendulum equations

$$\dot{x}_1 = x_2, \tag{18a}$$
$$\dot{x}_2 = -\sin(x_1) - u\cos(x_1), \tag{18b}$$

which describe frictionless motion of a pendulum mounted on a cart whose acceleration is u. The motion of the cart is not modeled; u is considered an input. We seek to design a controller that steers a sampled version of (18) from some neighborhood of the origin within a finite number of steps into the ellipsoid E defined by

$$E = (\pi, 0) + \left\{ x \in \mathbb{R}^2 \,\middle|\, \langle x | Hx \rangle \leq 1 \right\}, \quad H = \frac{1}{10\sqrt{2}} \begin{pmatrix} 13 & 3\sqrt{3} \\ 3\sqrt{3} & 7 \end{pmatrix} \tag{19}$$

and shown in Fig. 3(a), such that the closed loop satisfies the constraints

$$|x_2| \leq \pi, \tag{20a}$$
$$u \in \{0, -2, 2\}, \tag{20b}$$

with controls being constant on sampling intervals, i.e.,

$$U = \{[0,T] \to \mathbb{R} \colon t \mapsto 0, [0,T] \to \mathbb{R} \colon t \mapsto -2, [0,T] \to \mathbb{R} \colon t \mapsto 2\}$$

in the notation of section 2.

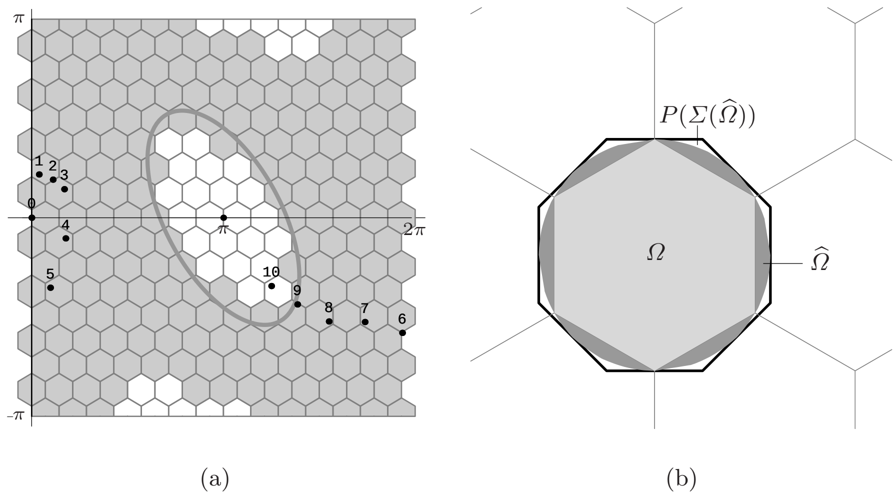

Fig. 3. (a) Covering of the state space by cells which defines the quantizer (5). A discrete controller obtained from a 2-complete abstraction of the behavior of the the sampled and quantized system (6) steers (6) from any shaded cell into some of the (unshaded) cells contained in the ellipsoid (19). The labeled points represent a particular closed-loop solution on the time interval $\{0, 1, \ldots, 10\}$. (b) A hexagon $\Omega \in C'$, its circumcircle $\widehat{\Omega}$, and a supporting polyhedral approximation $\Sigma(\widehat{\Omega})$ of $\widehat{\Omega}$.

To this end, let $\mathcal{U}$ be defined by (2), let φ denote the general solution of (18), and consider the sampled system (4) with sampling period $T = 0.35$. Define the quantizer Q of (5) by its system C of level sets ("cells"),

$$C = C' \cup \{\mathbb{R} \times]\pi, \infty[\,, \mathbb{R} \times]{-\infty}, -\pi[\},$$

where C' is a set of 238 translated and possibly truncated copies of the hexagon

$$\frac{\pi}{14\sqrt{3}} \operatorname{conv}\{(0, -2), (\sqrt{3}, -1), (\sqrt{3}, 1), (0, 2), (-\sqrt{3}, 1), (-\sqrt{3}, -1)\}, \tag{21}$$

see Fig. 3(a). Since the right hand side of (18) is periodic in x with period $(2\pi, 0)$, we tacitly consider the system (18) on the cylinder, so that C really is a covering of the state space of (18) and of (4).

For each $\Omega \in C'$, let $\widehat{\Omega}$ be a translated copy of the circumcircle of the hexagon (21), and $\Sigma(\widehat{\Omega})$, a supporting polyhedral approximation of $\widehat{\Omega}$ consisting of 8 equally distributed hyperplanes. See Fig. 3(b). The next result shows that the reachable set $\varphi(t, \widehat{\Omega}, u)$ is convex for all $\Omega \in C'$, all $u \in \mathcal{U}$, and all $t \in \{T, 2T\}$.

Table 1. Application of the algorithm proposed in section 4 to the present example (N: memory span of computed abstraction; s: number of half-spaces to be determined; q: number of polyhedra tested for emptiness; p: number of conservative polyhedral approximations of reachable sets to be stored).

N	s	q	p
0	1906	0	240
1	5184	30118	3060
2	21424	70496	22840

Theorem 6. *Let* $x_0 \in \mathbb{R}^2$, *u piecewise continuous with* $|u(\tau)| \leq \hat{u}$ *for all* $\tau \in \mathbb{R}_+$, $\omega = \left(1 + \hat{u}^2\right)^{1/4}$, $0 < t \leq \frac{\pi}{2\omega}$,

$$R = \frac{6\omega^2}{(1+\omega^2)^{3/2}\sinh(\omega t)\,(5+\cosh(2\omega t) - 10\exp(-\omega))}. \tag{22}$$

Then the reachable set $\varphi(t, \bar{B}(x_0, r), u)$ *is convex whenever* $0 < r \leq R$.

Indeed, the circumcircle of the hexagon (21) is of radius $\pi/(7\sqrt{3}) < 0.26$, while the bound (22) exceeds 0.26 for $\hat{u} = 2$ and $t \in \{T, 2T\}$, and $T < 2T < 1 < \frac{\pi}{2}\left(1 + \hat{u}^2\right)^{-1/4}$. Hence, for memory span $N \in \{0, 1, 2\}$, all relevant reachable sets are convex, and Theorem 3 guarantees that the algorithm proposed in section 4 yields a discrete abstraction B_N of the behavior of the sampled and quantized system (6).

We have implemented our algorithm from section 4 in *Mathematica 5.2* [33] and computed B_N for $N \in \{0, 1, 2\}$. Tab. 1 gives some statistics. Note that for $N = 2$, the number of half-spaces to be determined is less than the number of conservative polyhedral approximations of reachable sets to be stored, which demonstrates an important feature of our method. It took 0.8, 30.2 and 101.6 seconds to compute B_N for $N = 0, 1, 2$, respectively, on an IBM Thinkpad X60 with 1.83 GHz clock rate.

Based on the abstraction B_2 and using well-known methods from discrete mathematics [11,8], we have obtained a discrete controller which, by construction, steers the sampled and quantized system (6) from any cell shaded in Fig. 3 into some cell inside the ellipsoid E within at most 16 steps, and in particular, within 10 steps if starting from the origin. See Fig. 3(a). By construction, solutions of the closed loop remain in C' before entering E, which guarantees control and state constraints (20) are satisfied.

Acknowledgments. The author thanks Lars Grüne (Bayreuth) and Jan Willems (Leuven) for stimulating discussions, on a preliminary version of the contributions made in the present paper in the first case, and on certain properties of behaviors, in the second. The author also thanks Marcus von Lossow (Bayreuth) for the opportunity to use his implementation of shortest path algorithms for hypergraphs [34], and four anonymous referees whose constructive criticism helped to improve this text.

References

1. J. C. Willems. Models for dynamics. In *Dynamics reported, Vol. 2*, volume 2 of *Dynam. Report. Ser. Dynam. Systems Appl.*, pages 171–269. Wiley, 1989.
2. J. C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Automat. Control*, 36(3):259–294, 1991.
3. M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki. *Diagnosis and fault-tolerant control*. Springer, Berlin, 2003.
4. C. J. Tomlin, I. Mitchell, A. M. Bayen, and M. Oishi. Computational techniques for the verification of hybrid systems. *Proc. IEEE*, 91(7):986–1001, 2003.
5. X. D. Koutsoukos, P. J. Antsaklis, J. A. Stiver, and M. D. Lemmon. Supervisory control of hybrid systems. *Proc. IEEE*, 88(7):1026–1049, July 2000.
6. T. Moor and J. Raisch. Supervisory control of hybrid systems within a behavioural framework. *Systems Control Lett.*, 38(3):157–166, 1999. Hybrid control systems.
7. T. Moor, J. M. Davoren, and B. D. O. Anderson. Robust hybrid control from a behavioural perspective. In *Proc. 41th IEEE Conference on Decision and Control, Las Vegas, U.S.A., 2002*, pages 1169–1174, New York, 2002. IEEE.
8. L. Grüne and O. Junge. Approximately optimal nonlinear stabilization with preservation of the Lyapunov function property. In *Proc. 46th IEEE Conference on Decision and Control, New Orleans, Louisiana, U.S.A., 2007*, pages 702–707, New York, 2007. IEEE.
9. P. J. Ramadge and W. M. Wonham. Modular feedback logic for discrete event systems. *SIAM J. Control Optim.*, 25(5):1202–1218, 1987.
10. P. J. G. Ramadge and W. M. Wonham. The control of discrete event systems. *Proc. IEEE*, 77(1):81–98, Jan. 1989.
11. G. Gallo, G. Longo, S. Pallottino, and S. Nguyen. Directed hypergraphs and applications. *Discrete Appl. Math.*, 42(2-3):177–201, 1993. Combinatorial structures and algorithms.
12. A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability analysis. In N. A. Lynch and B. H. Krogh, editors, *Proc. 3rd Intl. Workshop Hybrid Systems: Computation and Control (HSCC), Pittsburgh, U.S.A., Mar. 23-25, 2000*, volume 1790 of *Lect. Notes Computer Science*, pages 202–214. Springer, 2000.
13. T. Moor and J. Raisch. Abstraction based supervisory controller synthesis for high order monotone continuous systems. In S. Engell, G. Frehse, and E. Schnieder, editors, *Modelling, Analysis, and Design of Hybrid Systems*, volume 279 of *Lect. Notes Control Inform. Sciences*, pages 247–265. Springer, 2002.
14. O. Junge. Rigorous discretization of subdivision techniques. In *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999)*, pages 916–918. World Sci. Publ., River Edge, NJ, 2000.
15. A. Puri, P. Varaiya, and V. Borkar. ϵ-approximation of differential inclusions. In *Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans, LA, U.S.A., 13-15 Dec. 1995*, volume 3, pages 2892–2897, New York, 1995. IEEE.
16. A. Chutinan and B. H. Krogh. Computational techniques for hybrid system verification. *IEEE Trans. Automat. Control*, 48(1):64–75, 2003.

17. S. Geist, G. Reißig, and J. Raisch. An approach to the computation of reachable sets of nonlinear dynamic systems – an important step in generating discrete abstractions of continuous systems. In S. Domek and R. Kaszyński, editors, *Proc. 11th IEEE Int. Conf. Methods and Models in Automation and Robotics (MMAR), Miedzyzdroje, Poland, Aug. 29-Sept. 1, 2005*, pages 101–106, 2005. Avail. at author's homepage.
18. L. Grüne and F. Müller. Set oriented optimal control using past information. In *Proc. 2008 Math. Th. of Networks and Systems (MTNS), Blacksburg, Virginia, U.S.A., July 28-Aug. 1*, 2008.
19. A. Tiwari. Abstractions for hybrid systems. *Form. Methods Syst. Des.*, 32(1):57–83, Feb. 2008. Proc. 7th Intl. Workshop Hybrid Systems: Computation and Control (HSCC), Philadelphia, U.S.A., Mar. 25-27, 2004.
20. M. Kloetzer and C. Belta. A fully automated framework for control of linear systems from temporal logic specifications. *IEEE Trans. Automat. Control*, 53(1):287–297, 2008.
21. P. Tabuada. An approximate simulation approach to symbolic control. *IEEE Trans. Automat. Control*, 53(6):1406–1418, 2008.
22. G. Reißig. Convexity of reachable sets of nonlinear ordinary differential equations. *Automat. Remote Control*, 68(9):1527–1543, Sept. 2007. Avail. at author's homepage. Russian transl. in Avtomat. i Telemekh., 2007, no. 9, pp. 64-78.
23. G. Reißig. Convexity of reachable sets of nonlinear discrete-time systems. In R. Kaszyński, editor, *Proc. 13th IEEE Int. Conf. Methods and Models in Automation and Robotics (MMAR), Szczecin, Poland, Aug. 27-30, 2007*, pages 199–204, 2007. Avail. at author's homepage.
24. M. W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*, volume 60 of *Pure and Appl. Math.* Academic Press, 1974.
25. P. Hartman. *Ordinary differential equations*, volume 38 of *Classics in Applied Mathematics*. Soc. Ind. Appl. Math. (SIAM), Philadelphia, PA, U.S.A., 2002.
26. R. Halin. *Graphentheorie*. Wiss. Buchgesellschaft, Darmstadt, 2nd edition, 1989.
27. R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., U.S.A., 1970.
28. G. Reißig. Local fill reduction techniques for sparse symmetric linear systems. *Electr. Eng.*, 89(8):639–652, Sept. 2007. Avail. at author's homepage.
29. G. Reißig. Fill reduction techniques for circuit simulation. *Electr. Eng.*, 90(2):143–146, Dec. 2007. Avail. at author's homepage.
30. G. Zampieri and G. Gorni. Local homeo- and diffeomorphisms: invertibility and convex image. *Bull. Austral. Math. Soc.*, 49(3):377–398, 1994.
31. B. T. Polyak. Convexity of nonlinear image of a small ball with applications to optimization. *Set-Valued Anal.*, 9(1-2):159–168, 2001. Wellposedness in optimization and related topics (Gargnano, 1999).
32. N. A. Bobylev, S. V. Emel′yanov, and S. K. Korovin. Convexity of images of convex sets under smooth maps. *Nelineinaya Dinamika i Upravlenie*, (2):23–32, 2002. (Russian. Engl. transl. in Comput. Math. Model. 15 (2004), no. 3, 213-222.).
33. S. Wolfram. *The Mathematica® book*. Wolfram Media, Inc., Champaign, IL, U.S.A., 5th edition, 2003.
34. M. von Lossow. A min-max version of Dijkstra's algorithm with application to perturbed optimal control problems. In *Proceedings in Applied Mathematics and Mechanics ICIAM07/GAMM07, Zürich, Schweiz, 2007*, volume 7, 2007.