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AN EXTENSION OF THE NORMAL TREE METHOD

Gunther Reißig

Technische Universität Dresden, Fakultät Elektrotechnik, Institut f. Regelungs- u. Steuerungstheorie, Mommsenstr. 13, 01062 Dresden, Germany, reiszig@erss11.et.tu-dresden.de

Abstract — **An extension of the normal tree method is presented. It applies to linear electrical networks that may contain independent and controlled sources, nullors, ideal transformers, and, under a topological restriction, gyrators, in addition to resistors, capacitors, and inductors. The proposed method relies on the concept of normal pair of conjugate trees and yields generically full sets of state coordinates. In addition, networks of generic index 1 are characterized and a systematic method of index-1-regularization is proposed.**

I. INTRODUCTION

The normal tree method was initially introduced by BRYANT in order to determine full sets of state coordinates of passive RLC networks [1]. It requires to construct a spanning tree that contains a maximal number of capacitor branches and a minimal number of inductor branches. The voltages of capacitors in that tree together with currents of inductors not in the tree may be taken as state coordinates.

Being purely topological, the normal tree method relys on the fact that properties of RLC networks, such as solvability and the number of state coordinates, do not depend on the specific choice of the network parameters, such as resistances, as long as these parameters are positive.

On one hand, properties of networks containing nullors or controlled sources do not only depend on the network topologies, but also on the network parameters. Therefore, it is impossible to give a purely topological method that yields, say, the number of state coordinates for such a network for a specific choice of its parameters. On the other hand, values of these parameters are usually known up to some numerical accuracy only. Hence, knowledge of properties of a network for a specific choice of its parameters is of restricted practical importance.

For example, the circuit equations of the network in Fig. 1 are of index 3 for $\alpha = 1$. (Inductance L and capacitance C are assumed to be nonzero; for definition of index see [2].) In particular, we have

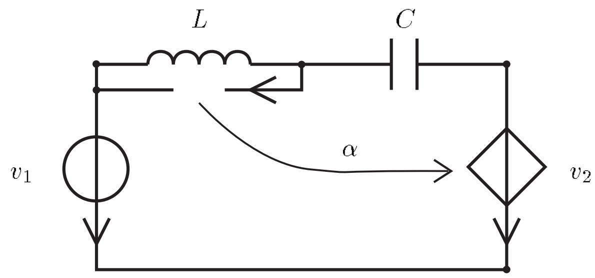

Figure 1: The circuit equations of the linear network shown are of index 3 if $\alpha = 1$ and otherwise of index 1.

$$\hat{v}_2(s) = -\hat{v}_1(s) s^2 LC,$$

where $\hat{}$ denotes Laplace transform, and hence, the second order derivative of the input voltage v_1 enters directly into the output voltage v_2. However, if the voltage controlled voltage source is a model of a real device, the value of its parameter α is not exactly known, and the result that the index will be 3 if $\alpha = 1$ is just useless. Indeed, for all values of the parameter α different from 1 the index is 1, we have

$$\hat{v}_2(s) = \hat{v}_1(s) \frac{s^2 \alpha LC}{s^2(\alpha - 1)LC - 1},$$

and the network does not show any differentiating behavior.

Unfortunately, it is in practice impossible to obtain properties of networks by symbolic computation, except for some small academic examples as that one above. To overcome that difficulty, one may check if networks have the property in question for all parameter values from an open dense subset of the space of all parameter values [3–7]. Such a property is called *generic* [8], which intuitively means the following: If a network shows that property for a specific parameter

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value, it still shows it after sufficiently small perturbations of that value ("open"). If, on the other hand, the network does not show the property in question for a specific parameter value, it will show it after arbitrarily small perturbations of that value ("dense").

In this paper, we present a purely topological extension of the normal tree method that applies to linear networks containing independent and controlled sources, nullors, ideal transformers and, under a topological restriction, gyrators, in addition to resistors, capacitors, and inductors. Our results rely on the concept of *pair of conjugate trees* [4, 9, 7], and our main result is as follows: The voltages of tree capacitors together with currents of co-tree inductors of a normal pair of conjugate trees determine a generically full set of state coordinates.

As a consequence of that result, we characterize networks with circuit equations of generic index 1, and a systematic method of index-1-regularization, i.e., a method of augmentation that yields circuit equations of generic index 1, is also presented.

We refer to [10] for proofs of our results, since these are too lengthy to be conveniently given here.

II. PRELIMINARIES

Matrix pencils and linear DAE's

Let $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be $n \times n$ matrices, $n \in \mathbb{N}$. The pair (A, B) is called *pencil* and is *regular* if there is some $s \in \mathbb{R}$ that $sA + B$ is regular. Otherwise it is *singular*. The pencil (A, B) is of *index* 1 if the degree in s of the polynomial $\det(sA + B)$ equals the rank of A.

A C^1-mapping $x\colon I \to \mathbb{R}^n$ is a *solution* of the Differential-Algebraic Equation (DAE)

$$A\dot{x}(t) + Bx(t) = q(t) \tag{1}$$

if it satisfies (1) for all $t \in I$ and $I \subseteq \mathbb{R}$ is an open, not necessarily bounded interval. A vector $x_0 \in \mathbb{R}^n$ is called *state* associated with t_0 for (1) if there is some solution x of (1) such that $x(t_0) = x_0$. (We assume throughout the paper that q is sufficiently smooth.)

Let (A, B) be regular. Consider a set $\mathcal{Z} = \{j_1, j_2, \ldots, j_r\} \subseteq \{1, 2, \ldots, n\}$ that contains r elements. We say that $\mathcal{Z}$ *determines a full set of state coordinates* of (1) if the components $x_{0,j_1}, x_{0,j_2}, \ldots, x_{0,j_r}$ of states x_0 may be chosen arbitrarily and uniquely determine the state.

Let now A and B be parameter dependent $n \times n$ matrices, i.e., $A, B\colon \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. The parameter dependent pencil (A, B) is *generically regular* and of *generic index* 1 if $(A(p), B(p))$ is regular and of index 1, respectively, for all p from some open dense subset of $\mathbb{R}^k$. We also say that $\mathcal{Z}$ *determines a generically full set of state coordinates* of the linear parameter dependent DAE

$$A(p)\dot{x}(t) + B(p)x(t) = q(t) \tag{2}$$

if $\mathcal{Z}$ determines a full set of state coordinates of (2) for all p from some open dense subset of $\mathbb{R}^k$.

Circuit equations of linear networks

Let $\mathcal{N}$ be a linear network that contains resistors, capacitors, inductors, ideal transformers, gyrators, nullors, and independent sources as well as the types of controlled sources shown in the first row of Table 1.

Let $Z = \{1, 2, \ldots, b\}$ be the branch set of $\mathcal{N}$, and denote by $p \in \mathbb{R}^k$ the vector containing all parameters of $\mathcal{N}$: Resistances, capacitances, inductances, gains of controlled sources, transformer and gyrator ratios. The circuit equations of $\mathcal{N}$ have the form

$$\underbrace{\begin{pmatrix} 0 \\ \hline 0 \\ \hline A_{LC}(p) \end{pmatrix}}_{A(p):=} \begin{pmatrix} v_1 \\ \vdots \\ v_b \\ i_1 \\ \vdots \\ i_b \end{pmatrix}^{\bullet} + \underbrace{\begin{pmatrix} K \\ \hline M(p) \\ \hline B_{LC} \end{pmatrix}}_{B(p):=} \begin{pmatrix} v_1 \\ \vdots \\ v_b \\ i_1 \\ \vdots \\ i_b \end{pmatrix} = q(t), \tag{3}$$

where $v_1, \ldots, v_b$ and $i_1, \ldots, i_b$ denote the branch voltages and currents, respectively, $M(p)$ corresponds to the voltage-current relations (VCR) of the resistive branches, and the submatrices $A_{LC}(p)$ and B_{LC} correspond to the VCR of the reactances. K corresponds to KIRCHHOFF's equations and consists of reduced circuit and cut set matrices. (Hence, $A(p)$ and $B(p)$ are $2b \times 2b$ matrices.)

We emphasize that each parameter p_j contained in the matrix $M(p)$, apart from transformer and gyrator ratios, occurs exactly once, and transformer and gyrator ratios occur exactly twice.

We call the network $\mathcal{N}$ *generically uniquely solvable* and of *generic index* 1 if the pencil (A, B) from (3) is generically regular and of generic index 1, respectively. Likewise, we say that $\mathcal{Z}$ *determines a generically full set of state coordinates* of $\mathcal{N}$ if $\mathcal{Z}$ determines a generically full set of state coordinates of (3).

Note that a generically uniquely solvable network $\mathcal{N}$ is of generic index 1 iff for all p from some open dense subset of $\mathbb{R}^k$, all capacitor voltages and all inductor currents may be given initial values independent from each other in order to obtain a unique solution.

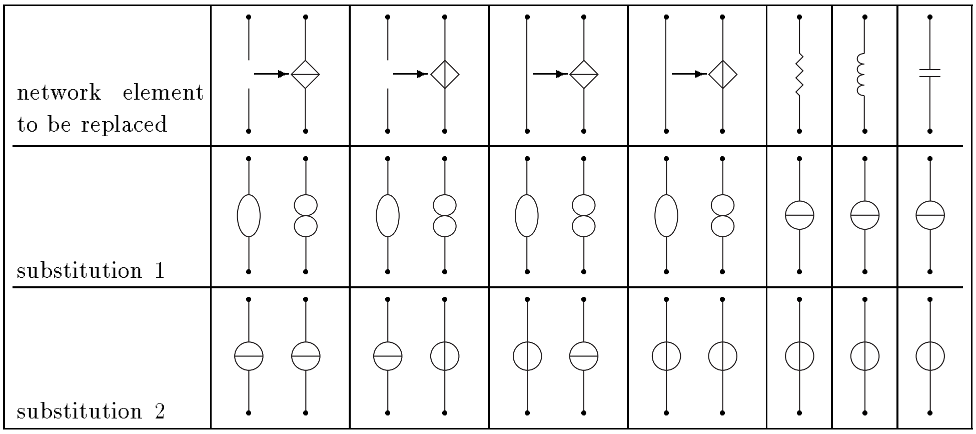

Table 1: Each network element in the first row is replaced by the elements in either the second or the third row of the same column.

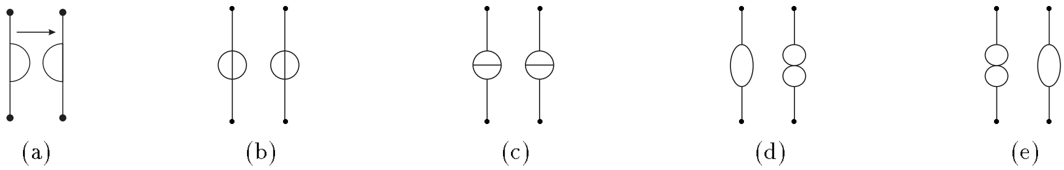

Figure 2: The gyrator of (a) is substituted by the elements in (b), (c), (d), or (e).

III. NORMAL PAIRS OF CONJUGATE TREES AND STATE COORDINATES

We first extend the term *pair of conjugate trees* [4,9,7] to networks that may contain reactances, and modify it in case the network contains gyrators. We then define *normal* pairs of conjugate trees in complete analogy to *normal* trees.

The main result of this paper is Theorem III.2, in which generically full sets of state coordinates are obtained from normal pairs of conjugate trees. Networks of generic index 1 are characterized in Corollary III.4, and a systematic method of augmentation that yields networks of generic index 1 is proposed in Corollary III.5. Finally, our statements are illustrated by Examples.

Throughout this paper, a *network* is a linear network that may contain resistors, capacitors, inductors, ideal transformers, gyrators, nullors, and independent sources as well as the types of controlled sources shown in the first row of Table 1.

III.1 Definition: *A pair* (t_1, t_2) *of spanning trees*[1] *of the graph of a network* $\mathcal{N}$ *is called a* pair of conjugate trees *if all gyrators, resistors, inductors, capacitors, and controlled sources of* $\mathcal{N}$ *can be replaced in accordance with Table 1 and Fig. 2 such that*

(i) t_1 *as well as* t_2 *contain all voltage source branches, no current source branch, and exactly one branch of each ideal transformer,*

(ii) t_1 *contains all norator branches and no nullator branch,*

(iii) t_2 *contains all nullator branches and no norator branch*

of the resulting network.

A pair of conjugate trees (t_1, t_2) *is called* normal *if*

$$|t_1 \cap C| + |L \setminus t_1| = \max\left\{ |w_1 \cap C| + |L \setminus w_1| \;\middle|\; (w_1, w_2) \text{ is a pair of conjugate trees of } \mathcal{N} \right\}$$

holds, i.e., if (t_1, t_2) *maximizes the sum of the number of tree capacitors and the number of co-tree inductors among all pairs of conjugate trees of the network* $\mathcal{N}$ *in question.* □

Note that, in general, t_1 contains other transformer branches than t_2.

III.2 Theorem: *Let the network* $\mathcal{N}$ *fulfill the following condition:*

[1] We use the terms *tree* and *forest* [6] synonymously.

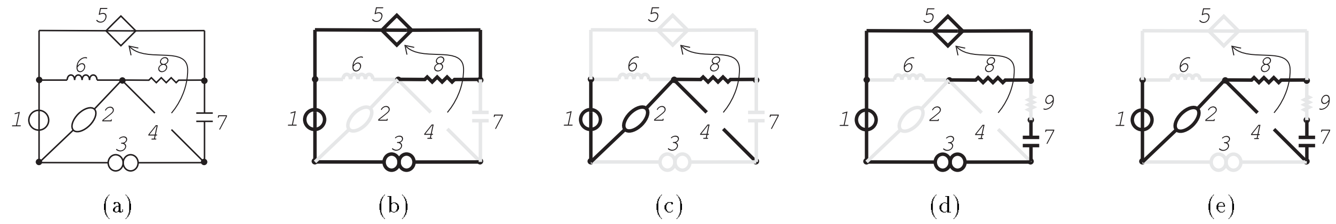

Figure 3: (a) Network investigated in Example III.3. A normal pair (t_1, t_2) of conjugate trees is illustrated: The branches of t_1 and t_2 are shown as thickened solid lines in (b) and (c), respectively. (d) and (e) show the network from (a) after modification according to Corollary III.5 together with trees t_1' and t_2', respectively, of a normal pair of conjugate trees (t_1', t_2') of the modified network.

(B) There is a partition of the branch set Z of the graph of $\mathcal{N}$ into disjoint subsets Z_1 and Z_2 such that B_1 through B_3 hold.

- *(B_1) If Z_c is the branch set of a circuit of the graph of $\mathcal{N}$, then either $Z_c \subseteq Z_1$ or $Z_c \subseteq Z_2$.*
- *(B_2) Let z_1 and z_2 be the branches of a gyrator of the network. Then either $z_1 \in Z_1$ and $z_2 \in Z_2$, or $z_1 \in Z_2$ and $z_2 \in Z_1$.*
- *(B_3) Let z_1 and z_2 be the branches of an ideal transformer of the network. Then either $\{z_1, z_2\} \subseteq Z_1$ or $\{z_1, z_2\} \subseteq Z_2$.*

Then $\mathcal{N}$ is generically uniquely solvable iff it has a pair of conjugate trees. Further, if (t_1, t_2) is a normal pair of conjugate trees of $\mathcal{N}$, then the voltages of capacitors in t_1 together with the currents of inductors not in t_1 determine a generically full set of state coordinates of $\mathcal{N}$. □

Note that condition (B) above is automatically fulfilled if the network in question does not contain gyrators. If the network does contain gyrators, condition (B) is often easy to check. For example, if the network consists of both electrical and mechanical components, that condition is fulfilled if for any ideal transformer, the two transformer branches are either both electrical, or both mechanical, and for any gyrator, exactly one branch is electrical. (The latter is actually fulfilled for a large class of electro-mechanical networks.)

Finally, note that one cannot overcome condition (B) by substituting gyrators by appropriate subnetworks containing, say, controlled sources. The parameters, e.g. gains of controlled sources, of such a subnetwork would not be independent from each other. Hence, Theorem III.2 will not apply to the resulting network.

III.3 Example: The network shown in Fig. 3(a) is a modification of the network from [7, Fig. 2.3]. It fulfills condition (B) in Theorem III.2, since it does not contain gyrators.

Since $(t_1, t_2) = (\{1, 3, 5, 8\}, \{1, 2, 4, 8\})$ is a pair of conjugate trees, that network is generically uniquely solvable. It is easy to see that the capacitor branch 7 cannot be contained in any pair of conjugate trees, and hence, the pair (t_1, t_2) is even normal. Thus, the current of the inductor branch 6 determines a generically full set of state coordinates. □

Of special interest is the case where *all* capacitor voltages and inductor currents represent state coordinates. The following results characterize that situation and also provide a systematic method to augment the network in order to obtain circuit equations of index 1.

III.4 Corollary: *Let the network $\mathcal{N}$ fulfill condition (B) in Theorem III.2 and let C and L be the sets of capacitor and inductor branches of $\mathcal{N}$, respectively.*
Then the following statements are equivalent.

- *(i) $\mathcal{N}$ is generically uniquely solvable and of generic index 1.*
- *(ii) The network obtained from $\mathcal{N}$ by substituting all capacitors and inductors by independent voltage and current sources, respectively, is generically uniquely solvable.*
- *(iii) The voltages of all capacitors together with the currents of all inductors of $\mathcal{N}$ determine a generically full set of state coordinates.*
- *(iv) $\mathcal{N}$ has a pair of conjugate trees (t_1, t_2) with $C \subseteq t_1$ and $L \cap t_1 = \emptyset$.* □

III.5 Corollary: *Let $\mathcal{N}$ be a generically uniquely solvable network that fulfills condition (B) in Theorem III.2. Let further (t_1, t_2) be a pair of conjugate trees of $\mathcal{N}$ and let $\mathcal{N}'$ be the network that is obtained from $\mathcal{N}$ as follows:*

- *(i) Augment $\mathcal{N}$ with either a resistor or an inductor in series to each capacitor not in t_1.*

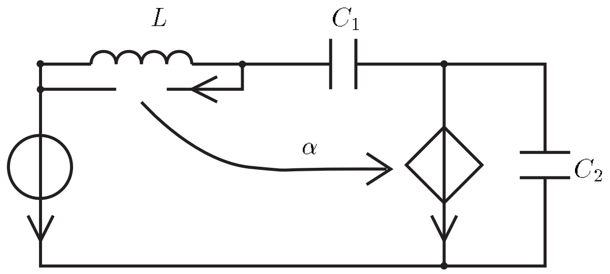

Figure 4: Network investigated in Example III.8. Its circuit equations are of index 4 if $\alpha = 1$ and otherwise of index 2.

(ii) Augment $\mathcal{N}$ with either a resistor or a capacitor in parallel to each inductor in t_1.

Then $\mathcal{N}'$ is generically uniquely solvable and of generic index 1. □

III.6 Example: The normal pair of conjugate trees of the network from Fig. 3(a) that was found in Example III.3 did not contain the capacitor branch 7. By Cor. III.4, that network is not of generic index 1. Fig. 3(d) and Fig. 3(e) show a modification of the network from Fig. 3(a) according to Cor. III.5 and illustrate a normal pair of conjuagte trees of the modified network. (Note that an inductor could have been inserted instead of the resistor.) □

III.7 Example: The network shown in Fig. 1 has a normal pair of conjugate trees (t, t), where t consists of the C-branch, the output branch of the controlled source, and the branch of the independent source. Hence, that network is of generic index 1, as already stated in the Introduction. □

III.8 Example: The network of Fig. 4 has a pair of conjugate trees (t, t), where t consists of the C_1-branch, the output branch of the controlled source, and the branch of the independent source. Since the C_2-branch cannot be in any pair of conjugate trees, the pair (t, t) is normal, and that network is not of generic index 1. Augmenting it with a resistor or an inductor in series to the C_2-branch yields a network of generic index 1. (Note that, if L, C_1, and C_2 are nonzero, the circuit equations (3) of the network shown in Fig. 4 are of index 2 iff $\alpha \neq 1$. Otherwise, they are of index 4.) □

IV. CONCLUSIONS

A necessary and sufficient condition for generic unique solvability of networks has been given, networks of generic index 1 have been characterized, generically full sets of state coordinates have been obtained, and a systematic method of augmentation that yields networks of generic index 1 has been proposed.

The results of this paper are applicable to linear electrical networks that may contain independent and controlled sources, nullors, ideal transformers and, under a topological restriction, gyrators, in addition to resistors, inductors, and capacitors.

Our main result, expressed in terms of pairs of conjugate trees, appears as a generalization of the well-known normal tree method to linear active networks. Although the latter had been extended in various directions earlier (see [11–15, 9, 5–7] and references cited therein), all these extensions impose serious restrictions on the network which concern the network topology as well as the class of admissible network elements. For example, the author does not know of any earlier result that finds state coordinates of networks that contain both nullors and ideal transformers.

Further research should be devoted to how far local properties of nonlinear networks can be obtained from generic properties of their linear counterparts.

V. ACKNOWLEDGEMENT

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