THE INDEX OF THE STANDARD CIRCUIT EQUATIONS OF PASSIVE RLCTG–NETWORKS DOES NOT EXCEED 2

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ABSTRACT

Linear and nonlinear networks composed of independent sources, resistive, capacitive and inductive subnetworks, as well as ideal transformers and gyrators (RLCTG–networks) are investigated. Branches may be arbitrarily coupled within the resistive, capacitive and inductive subnetworks and these subnetworks may be non–reciprocal. It is shown that the tractability index of the standard circuit equations of such networks does not exceed 2, provided that the resistance, capacitance, and inductance matrices of the subnetworks are positive definite.

1. INTRODUCTION

Many technical processes and systems, including electrical networks, may be modelled by Differential–Algebraic Equations (DAE's) of the type

$$F(x(t), \dot{x}(t), t) = 0. \tag{1}$$

In particular, circuit equations created within modern simulation programs are of form (1).

Knowing the *index* [1, 2] of (1) would be a great advantage for several reasons: Numerical solution of (1) is, as a rule, the more complicated the higher its index is. In case the index exceeds 2, typical problems to expect are extremely small stepsizes and simulation interrupts caused by exceptions in numerical routines, and choosing an appropriate numerical algorithm is easier if the index is a priori known [1].

Further, if (1) describes the dynamic behavior of an electrical circuit, the index determines how many input derivatives enter the solution, a question of high practical importance [3]. Therefore, there has been much effort at computing the index of linear [4–6] as well as nonlinear [7] DAE's.

To ask for the index of whole classes of DAE's is a much deeper and more difficult question, and for very few classes of electrical networks there have been given answers: Although the term 'index' is not used therein, one can conclude from the results of [8] that the index of the circuit equations of linear networks containing positive valued, uncoupled resistors and capacitors and coupled inductors having positive definite inductance matrices is either 1 or 2. Further, the index is 2 iff there is a capacitor–only loop or an inductor–only cut set. Later, the same was shown for the tableau equations of linear connected networks containing independent voltage sources and time–varying, uncoupled, positive valued resitors and capacitors [10]. Here, the index is two iff there is a loop consisting of voltage sources and capacitors only. Only recently, it was shown that the index of the MNA equations of nonlinear RLC networks containing coupled resistors, capacitors and inductors with positive definite resistance, capacity, and inductivity matrix, respectively, does not exceed 2, and an index–1–criterion has been given for these networks [11].

In this paper, we are concerned with linear as well as nonlinear *RLCTG–networks*, i.e., networks that are composed of independent sources, resistive, capacitive and inductive subnetworks, as well as ideal transformers and gyrators. We emphasize that branches may be arbitrarily coupled within the resistive, capacitive and inductive subnetworks and that we do *not* assume these subnetworks to be reciprocal. We show that the index of the standard circuit equations of such networks does not exceed 2, provided that the resistance, capacitance, and inductance matrices of the subnetworks are positive definite, and also give an index–1–criterion. The latter is different from that in [11] since the index of the standard circuit equations considered here may differ from that of the MNA equations considered in [11].

Our main contribution is that we include ideal transformers and gyrators as admissible network elements. This is not only of theoretical, but also of practical value: As a rule, networks describing systems composed of both electrical and mechanical parts contain transformers and gyrators. In that case, non of the known results on the index of the circuit equations are applicable. In view of what was said at the beginning on the importance of the index for the numerical behavior of DAE's, this is a serious lack of a priori information if such networks should be simulated.

Finally, we emphasize that our results hold for *all* networks having the required properties above. Even in the linear case, this is in contrast to approaches focussing on *generic* properties [4, 5], which yield results that hold only for *almost all* values of network parameters, e.g. resistances.

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2. THE LINEAR CASE

Let $\mathcal{N}$ be a linear RLCTG–network that consists of the elements shown in Table 1. The circuit equations of $\mathcal{N}$ may be stated in the form

$$\underbrace{\left(\begin{array}{c} 0 \\ \hline 0 \\ \hline A_{LC} \end{array}\right)}_{A:=} \begin{pmatrix} v_1 \\ \vdots \\ v_b \\ i_1 \\ \vdots \\ i_b \end{pmatrix}^{\bullet} + \underbrace{\left(\begin{array}{c} K \\ \hline M \\ \hline B_{LC} \end{array}\right)}_{B:=} \begin{pmatrix} v_1 \\ \vdots \\ v_b \\ i_1 \\ \vdots \\ i_b \end{pmatrix} = q(t), \qquad (2)$$

where $v_1, \ldots, v_b$ and $i_1, \ldots, i_b$ denote the branch voltages and currents, respectively, M corresponds to the voltage–current relations (VCR's) of the resistive branches, and the submatrices A_{LC} and B_{LC} correspond to the VCR's of the reactances. K corresponds to Kirchhoff's equations and consists of reduced circuit and cut set matrices.

Table 1 shows the admissible types of elements and also represents their VCR's in the same form as they enter the matrices M, A_{LC}, and B_{LC} of (2). (Throughout the paper we assume that q is of class C^1.)

By introducing a charge vector q and a flux vector φ, we can clearly substitute the VCR's

$$C\dot{v}_C - i_C = 0, \qquad L\dot{i}_L - v_L = 0$$

of capacitors and inductors

$$\begin{aligned} \dot{q} - i_C &= 0, & \dot{\varphi} - v_L &= 0, \\ q - Cv_C &= 0, & \varphi - L\dot{v}_L &= 0 \end{aligned}$$

and come to

$$\widetilde{A} \begin{pmatrix} q \\ \varphi \\ v \\ i \end{pmatrix}^{\bullet} + \widetilde{B} \begin{pmatrix} q \\ \varphi \\ v \\ i \end{pmatrix} = \tilde{q}(t). \qquad (3)$$

The latter are the *standard circuit equations* [9] of $\mathcal{N}$.

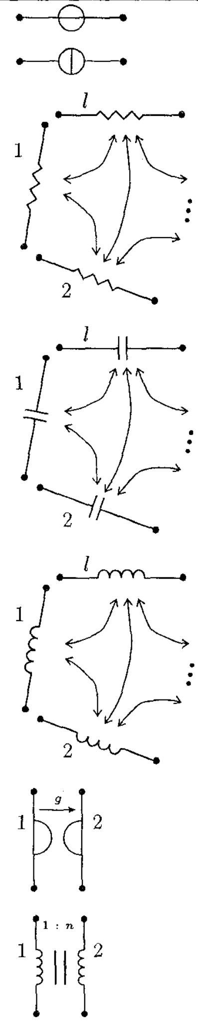
The network $\mathcal{N}$ is *uniquely solvable* if the pencil (A, B) from the circuit equations (2) is regular, i.e., if there is some $s \in \mathbb{R}$ such that $sA + B$ is regular. (Note that the pencil (A, B) from (2) is regular iff the pencil $(\widetilde{A}, \widetilde{B})$ from (3) is.)

For each regular pencil (A, B), there are regular matrices P and Q such that

$$A = P\,\mathrm{diag}(\mathrm{id}_r, N)Q \quad \text{and} \quad B = P\,\mathrm{diag}(W, \mathrm{id}_{n-r})Q,$$

where id_j is the $j \times j$ identity matrix for any j, $0 \leq r \leq n$, W is $r \times r$, and N is a nilpotent $(n-r)\times(n-r)$ matrix. The number r as well as the nilpotency index of N are uniquely determined by (A, B), and the latter is called *index* of the pencil (A, B), or equivalently, *index* of (2). Likewise, the index of the standard circuit equations (3) is the index of the pencil $(\widetilde{A}, \widetilde{B})$.

Our main result on linear RLCTG–networks is the following:



$v(t) = q(t)$

$i(t) = q(t)$

$R \cdot i - v = 0$

$C \cdot \dot{v} - i = 0$

$L \cdot \dot{i} - v = 0$

$v_1 + gi_2 = 0$
$v_2 - gi_1 = 0$

$nv_1 - v_2 = 0$
$i_1 + ni_2 = 0$

Table 1: Admissible network elements of linear RLCTG–networks and a representation of their voltage–current relations in the same form as they enter the matrices M, A_{LC}, and B_{LC} of (2). (In case of resistors, capacitors and inductors, the symbols v and i denote vectors and R, L and C denote matrices.)

2.1 Theorem: *Let $\mathcal{N}$ be a linear RLCTG–network that consists of the elements shown in Table 1. Assume that $\mathcal{N}$ is uniquely solvable and that the resistance, capacitance and inductance matrices of the resistive, capacitive and inductive, respectively, subnetworks contained in $\mathcal{N}$ are positive definite. Then the folowing hold:*

(i) *The index of the circuit equations (2) of $\mathcal{N}$ is either 1 or 2 and coincides with the index of the standard circuit equations (3) of $\mathcal{N}$.*

(ii) *The index of the circuit equations (2) of $\mathcal{N}$ is 1 iff the network that is obtained from $\mathcal{N}$ by replacing all capacitive branches by independent voltage sources and all inductive branches by independent current sources is uniquely solvable.*

(iii) *If the network $\mathcal{N}$ does neither contain gyrators nor ideal transformers, i.e., if $\mathcal{N}$ is an RLC–network, then the index of the circuit equations (2) of $\mathcal{N}$ is 1 iff there is some spanning forest containing all voltage sources, all capacitors, no current source and no inductor.* □

Proof (sketch): Let (2) be uniquely solvable. Then the index of (2) is 1 iff the matrix

$$A_1 = \begin{pmatrix} K \\ M \\ A_{LC} \end{pmatrix} \tag{4}$$

is regular, which immediately proves (ii). Assume now that A_1 is not regular. It can be shown that the index of (2) is 2 iff $Ax = 0 \wedge \exists_y\,(Bx = Ay \;\wedge\; By \in \operatorname{im} A)$ implies $x = 0$ [12]. For simplicity, write $x = (x^V, x^I)$ and

$$x^V = (x_V^V, x_I^V, x_R^V, x_L^V, x_C^V, x_T^V, x_G^V),$$
$$x^I = (x_V^I, x_I^I, x_R^I, x_L^I, x_C^I, x_T^I, x_G^I),$$

where x^V (resp. x^I) is the vector of branch voltages (resp. currents) of the network and x_V^V, x_I^V, x_R^V, x_L^V, x_C^V, x_T^V, x_G^V (resp. x_V^I, x_I^I, x_R^I, x_L^I, x_C^I, x_T^I, x_G^I) are the vectors of branch voltages (resp. currents) of the voltage sources, current sources, resistors, capacitors, inductors, ideal transformers, and gyrators.

By means of TELLEGEN's Theorem, we conclude from $Bx \in \operatorname{im} A$, $Ax = 0$ and positive definitness of the resistance, capacitance and inductance matrices that $x_C^U = 0$, $x_L^I = 0$, and $x_R^U = x_R^I = 0$.

From $Bx = Ay$, $By \in \operatorname{im} A$, TELLEGEN's Theorem, and positive definitness we conclude $x_C^I = 0$ and $x_L^U = 0$, which implies $B_{LC}x = 0$. From $A_1 x = 0$ we conclude $Ax = Bx = 0$, which implies $x = 0$ since (A, B) is regular.

It may be shown by the shuffle algorithm [13] that the indices of (A, B) and $(\widetilde{A}, \widetilde{B})$ coincide, which proves (i).

For the proof of (iii), note that A_1 is the coefficient matrix of the circuit equations of some linear passive R–network and that unique solvability of such a network depends on the network topology only. □

3. THE NONLINEAR CASE

We now consider nonlinear networks composed of independent sources, nonlinear resistive, capacitive and inductive subnetworks, and, although unusual, nonlinear transformers and gyrators. In analogy to the equations in Table 1, these nonlinear network elements are described as follows:

Independent voltage and current sources are described by

$$v(t) = q(t), \quad i(t) = q(t), \tag{5}$$

respectively. Resistive subnetworks are described by

$$R(i) - v = 0, \tag{6}$$

where R is a mapping $\mathbb{R}^l \to \mathbb{R}^l$ and $v = (v_1, \ldots, v_l)$ and $i = (i_1, \ldots, i_l)$. Capacitive and inductive subnetworks are described by

$$\dot{q} - i = 0, \quad q - Q(v) = 0, \tag{7}$$

and

$$\dot{\varphi} - v = 0, \quad \varphi - \Phi(i) = 0, \tag{8}$$

respectively, where $q = (q_1, \ldots, q_l)$ denotes the charge and $\varphi = (\varphi_1, \ldots, \varphi_l)$ is the flux. Q and Φ are mappings $\mathbb{R}^l \to \mathbb{R}^l$, and $Q'(v)$ and $\Phi'(i)$ are the (differential) capacitance and inductance, respectively. Ideal transformers are described by

$$n(v_1) - v_2 = 0, \quad i_1 + n(i_2) = 0, \tag{9}$$

where $n\colon \mathbb{R} \to \mathbb{R}$. The equations of the gyrator are

$$v_1 + g(i_2) = 0, \quad v_2 - g(i_1) = 0. \tag{10}$$

Equations (5)–(10) together with Kirchhoff's equations

$$K \cdot (v, i) = 0 \tag{11}$$

are the *standard circuit equations* [9] of nonlinear RLCTG–networks. These equations are of form

$$A\dot{x} - g(x) = q(t), \tag{12}$$

a special case of the DAE (1). Here, $x = (v, i, q, \varphi)$ and v is the vector of all branch voltages, i is the vector of all branch currents, q is the vector of all capacitor charges, and φ is the vector of all inductor fluxes.

In contrary to the linear case, there are several concepts of index for nonlinear DAE's. In this paper we rely on the concept of *tractability index* [14, 11]:

3.1 Definition: *Assume that g and q of (12) are defined on open sets $X \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}$, respectively, and that these mappings are of class C^1.*
Then the index *of (12) is 1 on some open set $V \subseteq X \times T$ if the index of $(A, g'(x))$ is 1 for all $(x, t) \in V$.*
The index *of (12) is 2 on some open set $V \subseteq X \times T$ if the index of $(A, g'(x))$ is 2 for all $(x, t) \in V$ and the dimension of*

$$\ker A \cap \{ y \in \mathbb{R}^n \mid g'(x)y \in \operatorname{im} A \}$$

is constant on V. □

3.2 Theorem: *Let $\mathcal{N}$ be a nonlinear RLCTG–network described by its standard circuit equations (12) consisting of the equations (5)–(11), and assume that g and q of (12) are of class C^1.*

Assume further that the linearization of $\mathcal{N}$ at some point $x_0 = (v_0, i_0, q_0, \varphi_0)$ fullfills the hypotheses of Theorem 2.1, i.e., let the pencil $(A, g'(x_0))$ be regular and let the differential resistance, capacitance and inductance matrices $R'(i_0)$, $Q'(v_0)$ and $\Phi'(i_0)$, respectively, be positive definite. Then the following hold:

(i) The index of the pencil $(A, g'(x_0))$ is either 1 or 2.

(ii) If the index of the pencil $(A, g'(x_0))$ is 1, then the index of the standard circuit equations (12) of $\mathcal{N}$ is 1 on some open neighborhood of x_0.

(iii) If the standard circuit equations (12) of $\mathcal{N}$ have an index on some open neighborhood of x_0, that index equals the index of the pencil $(A, g'(x_0))$.

(iv) If the network $\mathcal{N}$ does neither contain gyrators nor ideal transformers, i.e., if $\mathcal{N}$ is an RLC–network, then the standard circuit equations (12) of $\mathcal{N}$ have an index on some open neighborhood of x_0. That index is 1 iff there is some spanning forest containing all voltage sources, all capacitors, no current source and no inductor. □

Proof: (i) follows from the fact that $(A, g'(x_0))$ is the pencil of the standard circuit equations of some linear network that meets the requirements of Theorem 2.1, and (ii) and (iii) follow directly from Definition 3.1.

In order to show (iv), we perform one step of the shuffle algorithm and obtain some matrix $A_1(x)$ analogous to (4). The rank of $A_1(x)$ does not depend on x, since $A_1(x)$ is the coefficient matrix of the standard circuit equations of some passive R–network and the dimension of the solution space of such a network depends on the network topology only. □

4. CONCLUSIONS

It has been shown that the tractability index of the standard circuit equations of linear as well as nonlinear RLCTG–networks does not exceed 2, provided that the resistance, capacitance, and inductance matrices of the subnetworks are positive definite. We emphasize that branches may be arbitrarily coupled within the resistive, capacitive and inductive subnetworks and that we do not assume these subnetworks to be reciprocal.

Further, criteria for the standard circuit equations to be of index 1 have been given.

For networks that contain at least one ideal transformer or gyrator, the following problems remain open: Is there a *purely topological* criterion for the standard circuit equations to be of index 1? Does the tractability index necessarily exist under the hypotheses of Theorem 3.2?

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