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A NORMAL FORM FOR IMPLICIT DIFFERENTIAL EQUATIONS NEAR SINGULAR POINTS

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Abstract — **We consider differential equations $A(x)\dot{x} = g(x)$, where A is an $n \times n$-matrix of C^1-functions and g is C^1. We investigate the above differential equation about singular points x_0 that are standard in the sense of Rabier. In particular, the null space of $A(x_0)$ is of dimension 1. We show that there is a C^1-diffeomorphism that transforms the above equation about x_0 into $x_1\dot{x}_1 = \pm 1$, $\dot{x}_2 = \cdots = \dot{x}_n = 0$ about 0.**

I. INTRODUCTION

Many technical systems and processes may be modelled by implicit differential equations

$$A(x)\dot{x} = g(x), \tag{1}$$

where $A\colon U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \in C^1$ is a matrix of C^1-functions, $g\colon U \to \mathbb{R}^n \in C^1$, and $U \subseteq \mathbb{R}^n$ is open.

Whenever $A(x_0)$ is regular, the above *implicit* differential equation (1) is equivalent to the *explicit* ordinary differential equation

$$\dot{x} = A(x)^{-1} g(x) \tag{2}$$

in some neighborhood of x_0, and the usual existence and uniqueness results apply [1].

If A is not regular, these results do not apply, and the only known way to analyze (1) is to apply some index reduction method [2, 3, 4]. In many cases, that method yields another differential equation of form (1) with A regular everywhere.

In certain applications such as metal forming processes [5], fluid flows [6, 7], and nonlinear networks having impasse points [8], the matrix A of the differential equation (1) that is obtained by the above mentioned reduction method is neither regular nor of constant rank. It is exactly that latter case that is studied in the present paper.

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Rabier was the first to investigate the behavior of (1) for $n > 1$ in case A is not of constant rank and no further index reduction is feasible [9]. He considered *standard singular points*, i.e., points x_0 that fulfill the following conditions:

(S_1) $\dim \ker A(x_0) = 1$,

(S_2) $g(x_0) \notin \operatorname{im} A(x_0)$,

(S_3) $A'(x_0)kk \notin \operatorname{im} A(x_0)$ for all $k \in \ker A(x_0) \setminus \{0\}$.

(ker, im, and dim denote the null space, the image, and the dimension, respectively.) Under the additional assumptions $A \in C^2$ and $g \in C^2$ it was shown that there is some neighborhood $U' \subseteq U$ of x_0 such that the following holds:

(i) $S \cap U'$ is a C^1-submanifold of $\mathbb{R}^n$ of dimension $n-1$, where

$$S = \{x \in U \mid A \text{ is singular }\} \tag{3}$$

is the set of singular points of (1).

(ii) All points of $S \cap U'$ are standard singular points.

(iii) For any $x \in S \cap U'$ there are exactly two solutions that converge to x for $t \to 0$; no solution passes x.

(iv) The derivative of any solution that converges to a point $x \in S \cap U'$ blows up near x.

While a network exhibiting standard singular points will be given in Section III, let us first consider the simple differential equation

$$\begin{aligned} x_1\dot{x}_1 &= \sigma, \\ \dot{x}_2 &= 0, \\ &\vdots \\ \dot{x}_n &= 0 \end{aligned} \tag{4}$$

with $\sigma \in \{1, -1\}$ in some neighborhood of 0. The set of singular points is the hypersurface defined by $x_1 = 0$, all singular points are standard, and (i) and (ii) above hold with $U = U' = \mathbb{R}^n$. (See Fig. 1(a) for illustration.)

For $\sigma = -1$ and any $x_{0,2}, \ldots, x_{0,n}$,

$$x_1(t) = \pm\sqrt{-2t},$$
$$x_2(t) = x_{0,2}, \quad (5)$$
$$\vdots$$
$$x_n(t) = x_{0,n} \quad (6)$$

defines exactly two solutions for $t < 0$, and for $\sigma = 1$ and any $x_{0,2}, \ldots, x_{0,n}$,

$$x_1(t) = \pm\sqrt{2t}$$

together with (5) ... (6) also defines exactly two solutions for $t > 0$. No other solutions exist, except translations of the above solutions in time, and hence, (iii) and (iv) above also hold here.

It is the objective of the present paper to show that *any* differential equation of form (1) considered in some neighborhood of a standard singular point may be transformed by a C^1-diffeomorphism into (4) in some neighborhood of 0. That result together with a sketch of its proof is stated in Section II, and Section III contains an application.

II. MAIN RESULT

II.1 Theorem: *Let x_0 be a standard singular point of (1). Let further $k \in \ker A(x_0) \setminus \{0\}$, $u \in (\operatorname{im} A(x_0))^\perp \setminus \{0\}$, and*

$$\sigma = \operatorname{sign}\left(\langle u|g(x_0)\rangle \cdot \langle u|A'(x_0)kk\rangle\right),$$

where $\langle\cdot|\cdot\rangle$ is the inner product in $\mathbb{R}^n$ and $(\operatorname{im} A(x_0))^\perp$ is the orthogonal complement of $\operatorname{im} A(x_0)$.

Then there are open neighborhoods U' and $\tilde{U}'$ of x_0 and 0, respectively, and a C^1-diffeomorphism $\Sigma\colon U' \to \tilde{U}'$ such that a mapping x with values in U' is a solution of (1) iff $\Sigma \circ x$ is a solution of (4). □

Proof (sketch): Our hypotheses imply that σ is nonzero and independent of k and u. We assume without loss of generality $x_0 = 0$, $u = (\operatorname{adj} A(0))^T k$ and $k = \operatorname{adj}(A(0))g(0) = (1, 0, \ldots, 0)$, which could be achieved by an affine coordinate transformation that does not change σ. For convenience, we set $f(x) = \det A(x)$.

The first trick of our proof is to consider the differential equation

$$f(x)\dot{x} = \operatorname{adj}(A(x))g(x) \quad (7)$$

rather than (1), where adj denotes the transpose of the matrix of cofactors. Then x_0 is a standard singular point of (1) iff $f(x_0) = 0$ and

$$0 \neq f'(x_0)\operatorname{adj}(A(x_0))g(x_0). \quad (8)$$

Further, in some neighborhood of x_0, (7) has exactly the same solutions as (1) [9].

The second trick is to apply the Straightening Out Theorem [1, Satz 19.1] to

$$\dot{x} = \operatorname{adj}(A(x))g(x). \quad (9)$$

This yields a diffeomorphism Ψ with $\Psi'(0) = \mathrm{id}$ (id is the identity mapping) that transforms (7) into

$$\begin{aligned} f(\Psi(x))\dot{x}_1 &= 1, \\ \dot{x}_2 &= 0, \\ &\vdots \\ \dot{x}_n &= 0. \end{aligned} \quad (10)$$

(Trajectories of (10) are illustrated in Fig. 1(b).)

Next, the set S is straightened: (8) implies $D_1(f \circ \Psi)(0) \neq 0$. Hence, by the Implicit Function Theorem, there is a C^1-mapping η defined on some neighborhood of $0 \in \mathbb{R}^{n-1}$ such that $f(\Psi(x)) = 0$ iff $x_1 = \eta(x_2, \ldots, x_n)$. Obviously then, the mapping

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1 - \eta(x_2, \ldots, x_n), x_2, \ldots, x_n)$$

is a local diffeomorphism; we denote its inverse by Φ. The diffeomorphism Φ transforms (10) into

$$\begin{aligned} \tilde{f}(x)\dot{x}_1 &= \sigma, \\ \dot{x}_2 &= 0, \\ &\vdots \\ \dot{x}_n &= 0, \end{aligned} \quad (11)$$

where we have set $\tilde{f}(x) = \sigma f(\Psi(\Phi(x)))$. Now, in some neighborhood of 0, $\tilde{f}(x) = 0$ iff $x_1 = 0$. (Trajectories of (11) are illustrated in Fig. 1(c).)

The last step of the proof is to show that

$$\theta(t,p) = \operatorname{sign}(t)\sqrt{2\int_0^t \tilde{f}(\tau,p)d\tau}$$

is a solution of the differential equation

$$\theta\dot{\theta} = \tilde{f}(t,p), \quad (12)$$

where $\dot{\theta}$ means $D_1\theta$ and $p \in \mathbb{R}^{n-1}$ is a parameter. We will show that $\theta \in C^1$; it is then obvious that (12) is fulfilled:

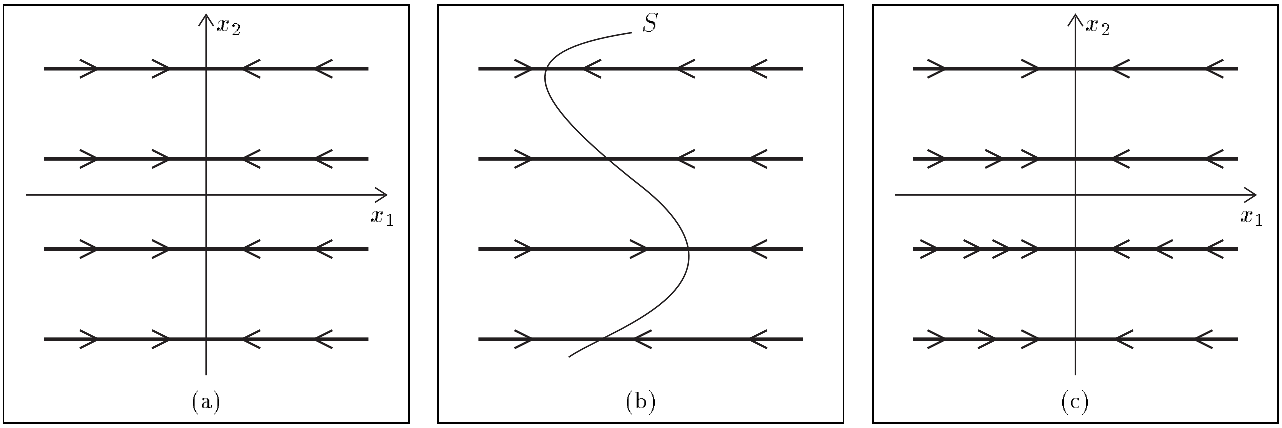

Figure 1: Illustration of trajectories in case $n = 2$ and $\sigma = -1$. (a) Trajectories of (4). (b),(c) Trajectories of (10) and (11), respectively.

First, one can show $\mathrm{sign}(D_1 f(0)) = \sigma$, which implies $D_1\tilde{f}(0) = \sigma^2 = 1 > 0$, and hence, the definition of θ makes sense. The Taylor formula

$$\tilde{f}(t,p) = \int_0^1 D_1\tilde{f}(\zeta t, p)t \; d\zeta \tag{13}$$

yields

$$|\theta(t,p)| \geq M_0|t| \tag{14}$$

for some $M_0 > 0$ and all t and p sufficiently small. Obviously, $\theta(0,\cdot) = 0$, and for $t \neq 0$ we have

$$D_1\theta(t,p) = |\tilde{f}(t,p)| \cdot |\theta(t,p)|^{-1}$$

and

$$D_2\theta(t,p) = \mathrm{sign}(t) \int_0^t D_2\tilde{f}(\tau,p)\; d\tau \cdot |\theta(t,p)|^{-1}.$$

Let now $(0,p_0) \in U$ and $\varepsilon > 0$. From (13) we obtain

$$\left|\tilde{f}(t,p) - D_1\tilde{f}(0,p_0)t\right| \leq \varepsilon|t|$$

and

$$\left|\int_0^t \tilde{f}(\tau,p)\; d\tau - \frac{1}{2}D_1\tilde{f}(0,p_0)t^2\right| \leq \varepsilon t^2$$

for all t and $p - p_0$ sufficiently small, which yields

$$\lim_{t\to 0}\left(\frac{2}{t^2}\int_0^t \tilde{f}(\tau,p_0)\; d\tau\right) = D_1\tilde{f}(0,p_0),$$

and hence $D_1\theta(0,p_0) = \left(D_1\tilde{f}(0,p_0)\right)^{1/2}$.

Further, from $D_2\tilde{f}(0,p_0) = 0$ and continuity of $D_2\tilde{f}$, we obtain

$$\left|\int_0^t D_2\tilde{f}(\tau,p)h\; d\tau\right| \leq \varepsilon|t|\;||h|| \tag{15}$$

for sufficiently small t and $p - p_0$. (15) and (14) yield $|D_2\theta(t,p)h| \leq M_1\varepsilon||h||$ for sufficiently small t and $p - p_0$ and some $M_1 > 0$, which implies continuity of $D_2\theta$ at the point $(0,p_0)$.

It remains to show that $D_1\theta$ is continuous at $(0,p_0)$: For t and $p - p_0$ sufficiently small and $t \neq 0$ we have

$$\begin{aligned}
|D_1\theta(t,p) - D_1\theta(0,p_0)| &= \left|\frac{|\tilde{f}(t,p)|}{|\theta(t,p)|} - D_1\theta(0,p_0)\right| \\
&\leq \frac{\left||\tilde{f}(t,p)| - D_1\tilde{f}(0,p_0)|t|\right|}{|\theta(t,p)|} \\
&\qquad + \left|\frac{D_1\tilde{f}(0,p_0)|t|}{|\theta(t,p)|} - D_1\theta(0,p_0)\right| \\
&\leq \frac{\varepsilon|t|}{M_0|t|} + D_1\theta(0,p_0)\left|\frac{\left(D_1\tilde{f}(0,p_0)t^2\right)^{1/2}}{\left(2\int_0^t \tilde{f}(\tau,p)\; d\tau\right)^{1/2}} - 1\right| \\
&\leq \frac{\varepsilon}{M_0} + D_1\theta(0,p_0)\frac{\left|D_1\tilde{f}(0,p_0)t^2 - 2\int_0^t \tilde{f}(\tau,p)\; d\tau\right|}{2\int_0^t \tilde{f}(\tau,p)\; d\tau} \\
&\leq \frac{\varepsilon}{M_0} + D_1\theta(0,p_0)\frac{2\varepsilon t^2}{M_0^2 t^2}.
\end{aligned}$$

Since $D_1\theta(0,0) > 0$, the mapping Θ defined by

$$\Theta(x_1,x_2,\ldots,x_n) = (\theta(x_1,x_2,\ldots,x_n), x_2,\ldots,x_n)$$

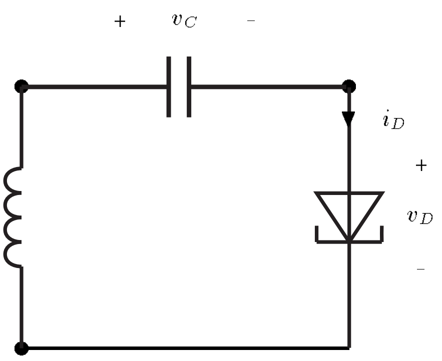

Figure 2: Network investigated in Section III.

is a local diffeomorphism. It is easily shown that Θ eventually transforms (11) into (4), since θ is a solution to the differential equation (12). The diffeomorphism Σ sought is $(\Psi \circ \Phi \circ \Theta^{-1})^{-1}$. $\square$

III. EXAMPLE

The network from [8, Example IV.4.], shown in Fig. 2, contains a capacitor of capacitance 1, an inductor of inductance 1, and a tunnel diode. It is described by the equations

$$\dot{v}_C = i_D, \tag{16}$$

$$\dot{i}_D = -v_C - v_D, \tag{17}$$

$$0 = i_D - v_D^3 + 9v_D^2 - 24v_D, \tag{18}$$

where (18) represents the voltage current relation of the tunnel diode.

In this Example, an index reduction may be performed as follows: By differentiating (18) we see that any solution of (16)-(18) fulfills

$$\dot{v}_C = v_D^3 - 9v_D^2 + 24v_D \tag{19}$$

$$3\dot{v}_D(6v_D - v_D^2 - 8) = v_C + v_D. \tag{20}$$

Conversely, if (v_D, i_D) is a solution of (19)-(20), then (v_C, v_D, i_D) with

$$i_D(t) = v_D(t)^3 - 9v_D(t)^2 + 24v_D(t)$$

is a solution of (16)-(18), and hence, the system (19)-(20) completely describes the behavior of (16)-(18).

If we write (19)-(20) in the form (1) we have

$$A(v_C, v_D) = \begin{pmatrix} 1 & 0 \\ 0 & 3(6v_D - v_D^2 - 8) \end{pmatrix}$$

and

$$g(v_C, v_D) = \begin{pmatrix} v_D^3 - 9v_D^2 + 24v_D \\ v_C + v_D \end{pmatrix}.$$

Obviously, $A(v_C, v_D)$ is singular iff $v_D \in \{2, 4\}$, i.e., the set of singular points is

$$S = \mathbb{R} \times \{2, 4\}.$$

Further, we have

$$\begin{aligned} \ker A(v_C, v_D) &= \{0\} \times \mathbb{R}, \\ \operatorname{im} A(v_C, v_D) &= \mathbb{R} \times \{0\}, \\ A'(v_C, v_D)kk &= \begin{pmatrix} 0 & 0 \\ 0 & 3(6 - 2v_D) \end{pmatrix} k \\ &= \begin{pmatrix} 0 \\ 6(3 - v_D) \end{pmatrix} \notin \operatorname{im} A(v_C, v_D), \end{aligned}$$

and σ equals the sign of

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} v_D^3 - 9v_D^2 + 24v_D \\ v_C + v_D \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 6(3 - v_D) \end{pmatrix} \right\rangle,$$

that is,

$$\sigma = -\operatorname{sign}\left((v_C + v_D)(3 - v_D)\right) \tag{21}$$

for $k = \binom{0}{1}$ and $u = \binom{0}{1}$ and all $(v_C, v_D) \in S \setminus \{(-2, 2), (-4, 4)\}$. Hence, all singular points except $(-2, 2)$ and $(-4, 4)$ are standard. (The latter are points of the set N_ψ of [10].) In particular, it follows from (21) and Theorem II.1 that exactly two solutions converge at finite time to any point (v_C, v_D) with $v_D = 2$ and $v_C \neq -2$, where convergence is for increasing time if $v_C < -2$ and for decreasing time if $v_C > -2$. This is illustrated in Fig. 3.

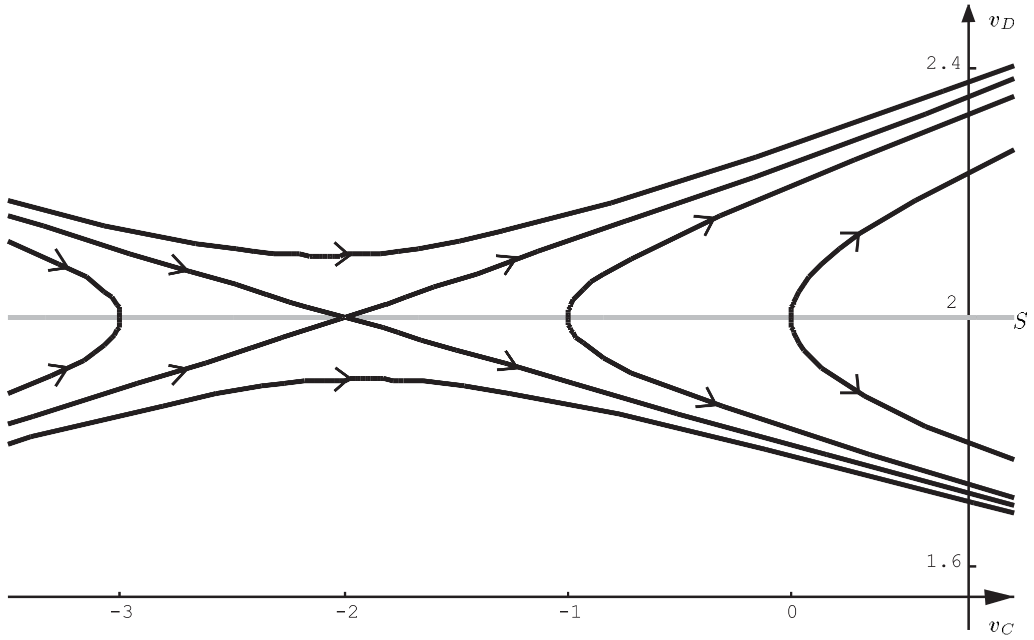

Figure 3: Trajectories of the system (19)-(20).

IV. CONCLUSIONS

It has been shown that any implicit differential equation $A(x)\dot{x} = g(x)$ about a standard singular point in the sense of RABIER may be transformed by a C^1-diffeomorphism into the normal form $x_1\dot{x}_1 = \pm 1$, $\dot{x}_2 = \cdots = \dot{x}_n = 0$ about 0. This result is new and provides a deeper understanding of sytems' behavior near standard singular points. In particular, the results of [9] follow from the above normal form.

Our approach does not work for singular points that are not standard. For example, the system (19)-(20) about the points $(-2, 2)$ and $(-4, 4)$ cannot be transformed into our normal form. The importance of standard singular points is that the fact that a trajectory eventually meets such a point is not affected by small perturbations of either the initial value or the system itself.

References

[1] H. Amann. *Gewöhnliche Differentialgleichungen.* Walter de Gruyter, Berlin and New York, 1983. de-Gruyter-Lehrbuch.

[2] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, volume 14 of *Classics in Applied Mathematics.* 1995.

[3] A. Szatkowski. Generalized dynamical systems: differentiable dynamic complexes and differential dynamic systems. *Internat. J. Systems Sci.*, 21(8):1631–1657, 1990.

[4] P. J. Rabier and W. C. Rheinboldt. A geometric treatment of implicit differential-algebraic equations. *J. Differential Equations*, 109:110–146, 1994.

[5] C. A. Hall, X. Lei, and P. J. Rabier. A nonstandard symmetry breaking phenomenon in sheet metal stretching. *Internat. J. Engrg. Sci.*, 32(9):1381–1397, 1994.

[6] T. A. Porsching. A network model for two-fluid, two-phase flow. *Num. Meth. Part. Diff. Equ.*, 1:295–313, 1985.

[7] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems*, volume 14 of *Computational Mathematics.* Springer-Verlag, 1991.

[8] G. Reißig. Differential-algebraic equations and impasse points. *IEEE Trans. Circuits and Systems Part I*, 43(2):122–133, Feb. 1996.

[9] P. J. Rabier. Implicit differential equations near a singular point. *J. Math. Anal. Appl.*, 144(2):425–449, Dec. 1989.

[10] V. Venkatasubramanian, H. Schättler, and J. Zaborszky. A taxonomy of the dynamics of the large power system with emphasis on its voltage stability. In *Proc. of the NSF Int. Workshop on Bulk Power System Voltage Phenomena-II*, pages 9–52, Aug. 1991.