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STRAIGHTENING OUT TRAJECTORIES NEAR SINGULAR POINTS OF IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract

Under appropriate assumptions, it is shown that there is a diffeomorphism that transforms solutions of the implicit differential equation

$$A(x)\dot{x} = g(x) \tag{1}$$

near points at which A is singular into solutions of the normal form

$$\begin{aligned} x_1^r \dot{x}_1 &= \sigma, \\ \dot{x}_2 = 0, \; \ldots, \; \dot{x}_n &= 0 \end{aligned} \tag{2}$$

near 0, and vice versa, where $\sigma = \pm 1 = \text{const}$. In particular, standard singular points in the sense of Rabier correspond to $r = 1$ in (2). A practical example leading to the normal form (2) with $r = 2$ is also given.

1 Introduction

Many technical systems and processes may be modelled by implicit differential equations (1), where $A\colon U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \in C^1$ is a matrix of C^1–functions, $g\colon U \to \mathbb{R}^n \in C^1$, $U \subseteq \mathbb{R}^n$ is open, and $n \in \mathbb{N}$. Whenever A is regular at some point $x_0 \in U$, the above *implicit* differential equation (1) is locally equivalent to the *explicit* ordinary differential equation $\dot{x} = A(x)^{-1}g(x)$ in some neighborhood of x_0, and the usual existence, uniqueness and smoothness results apply.

If $A(x_0)$ is not regular, these results do not apply, and the analysis of (1) near x_0 becomes more difficult. For example, it may happen that the set of points at which A is singular forms an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$ containing x_0. This case, which is the concern of the present paper, was first studied by Rabier for $n > 1$ [Rab89]. He considered *standard singular points*, i.e., points x_0 that fulfill the following conditions:

(S_1) $\dim \ker A(x_0) = 1$,

(S_2) $g(x_0) \notin \operatorname{im} A(x_0)$,

(S_3) $A'(x_0)kk \notin \operatorname{im} A(x_0)$ for all $k \in \ker A(x_0) \setminus \{0\}$.

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(ker, im, and dim denote the null space, the image, and the dimension, respectively, $A'(x_0)$ denotes the derivative at x_0 of the mapping $A\colon U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ from (1).)

Under the above assumptions, it was shown in [Rab89] that the set of points at which A is singular forms an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$ locally near x_0. Further, σ defined by

$$\sigma = \operatorname{sign}\left(\langle u|g(x_0)\rangle \cdot \langle u|A'(x_0)kk\rangle\right) \tag{3}$$

is nonzero and does neither depend on k, nor on u, provided $k \in \ker A(x_0) \setminus \{0\}$ and $u \in (\operatorname{im} A(x_0))^{\perp} \setminus \{0\}$. (Here, $^{\perp}$ denotes the orthogonal complement and $\langle \cdot|\cdot \rangle$ is the usual inner product in $\mathbb{R}^n$.)

If $\sigma = -1$, the standard singular point x_0 is called *attracting*, and if $\sigma = 1$, it is called *repelling*. It was also shown that the set of standard singular points forms an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$, that no solution passes x_0, and that all solutions are transversal to the submanifold of standard singular points in the sense of [Rab89].

Rabier reduced the analysis of (1) to that of

$$f(x)\dot{x} = G(x), \tag{4}$$

where $f(x) = \det A(x)$ and $G(x) = (\operatorname{adj} A(x))g(x)$. ($\operatorname{adj} A(x)$ is the transpose of the matrix of cofactors of $A(x)$.) Indeed, one can show the following:

1.1 Proposition: *Let $f(x) = \det A(x)$ and $G(x) = (\operatorname{adj} A(x))g(x)$. If ($S_1$) and ($S_2$) hold, or if $f(x_0) = 0$ and $G(x_0) \neq 0$ hold, then (1) and (4) have exactly the same solutions near x_0.* □

Under the additional assumptions $A \in C^2$ and $g \in C^2$ Rabier has shown the following for attracting (resp. repelling) standard singular points x_0:

(1) There are some $T > 0$ and exactly two solutions ξ_1 and ξ_2 of (1), defined on $]-T, 0[$ (resp. $]0, T[$), such that $\lim\limits_{t\to 0} \xi_1(t) = \lim\limits_{t\to 0} \xi_2(t) = x_0$.

(2) The derivatives of solutions blow up near x_0, i.e., $\lim\limits_{t\to 0} \|\dot{\xi}_1(t)\| = \lim\limits_{t\to 0} \|\dot{\xi}_2(t)\| = \infty$.

Following [Rab89], Medved' investigated the implicit differential equation (4) in order to analyze (1) [Med91]. Under the assumptions $f\colon \mathbb{R}^n \to \mathbb{R} \in C^\infty$, $G\colon \mathbb{R}^n \to \mathbb{R}^n \in C^\infty$, $f(0) = 0$ and $G(0) \neq 0$, the following was shown: For all f from some residual (generic) subset of $C^\infty(\mathbb{R}^n, \mathbb{R})$ there is some $\alpha\colon U \to \mathbb{R} \setminus \{0\} \in C^\infty$ such that (4) is C^∞–conjugate near 0 to

$$\begin{aligned} x_1\dot{x}_1 &= \alpha(x), \\ \dot{x}_2 = 0, \ &\ldots, \dot{x}_n = 0 \end{aligned} \tag{5}$$

near 0. (A differential equation is C^r*–conjugate* near x_0 to another differential equation near $\tilde{x}_0$ if there are open neighborhoods U' and $\widetilde{U}'$ of x_0 and $\tilde{x}_0$, respectively, and some C^r–diffeomorphism $\Phi\colon U' \to \widetilde{U}'$ with $\Phi(x_0) = \tilde{x}_0$ such that a curve x with values in U' is a solution of the first equation iff the transformed curve $\Phi \circ x$ is a solution of the second.)

An investigation of Medved's proof shows that, for the local C^∞–conjugacy of (4) and (5), it suffices to assume $f'(0)G(0) \neq 0$, which is, apart from stronger smoothness requirements, equivalent to say that 0 is a standard singular point of (1).

MEDVED' also investigated the case $f'(0)G(0) = 0$: Under appropriate assumptions on higher order directional (or LIE) derivatives of f along G it was shown in [Med91] that (4) is C^∞–conjugate near 0 to

$$\left(x_1^r + \sum_{j=0}^{r-1} x_1^j \beta_j(x_2, \ldots, x_n) \right) \dot{x}_1 = \alpha(x), \qquad \dot{x}_2 = 0, \; \ldots, \dot{x}_n = 0 \tag{6}$$

near 0, where $\alpha\colon U \to \mathbb{R} \setminus \{0\} \in C^\infty$, $\beta_j\colon \mathbb{R}^{n-1} \to \mathbb{R} \in C^\infty$ and $\beta_j(0) = 0$ for all $j \in \{0, 1, \ldots, r-1\}$.

Recently, REIẞIG and BOCHE sharpened the first result of MEDVED' as follows: If A and g of (1) are of class C^1 and if x_0 is a standard singular point, then (1) is C^1–conjugate near x_0 to (5) near 0 with either $\alpha = 1 = $ const. or $\alpha = -1 = $ const. [RB97].

In Section 2 of this paper, we show under appropriate assumptions that the implicit differential equation (1) is C^s–conjugate near points at which A is singular to the normal form (2) near 0 for some s and some r. In particular, standard singular points correspond to $r = 1$ in (2). A practical example leading to (2) with $r = 2$ is given in Section 3.

2 Main result

2.1 Theorem: *Let $n, r \in \mathbb{N}$, $s \in \mathbb{N} \cup \{\infty, \omega\}$, and let $\mu = r + s - 1$ if $s \in \mathbb{N}$ and $\mu = s$ otherwise. Let further be $U \subseteq \mathbb{R}^n$ an open neighborhood of x_0, $f\colon U \to \mathbb{R} \in C^\mu$, $G\colon U \to \mathbb{R}^n \in C^\mu$ and*

$$\sigma = \operatorname{sign}\left(D^r f(x_0) G(x_0)^r \right). \tag{7}$$

(Here, C^ω is the class of analytic mappings and $D^r f$ is the r–th order derivative of f.)

Assume further that $f(x_0) = 0$, $\sigma \neq 0$, and that $f^{-1}(0)$ is an $(n-1)$–dimensional C^s–submanifold of $\mathbb{R}^n$, and $D^j f(x) = 0$ holds for all j with $1 \leq j < r$ and for all $x \in f^{-1}(0)$.

Then the differential equation (4) is C^s–conjugate near x_0 to the normal form (2) near 0. □

Note that, if in addition to the hypotheses of the above Theorem, the mappings f and G are defined by $f(x) = \det A(x)$ and $G(x) = (\operatorname{adj} A(x))g(x)$, then the differential equation (1) is C^s–conjugate near x_0 to the normal form (2) near 0.

Proof: We refer to [Rei97] for a complete proof and give a sketch only: First, assume $x_0 = 0$ and $G(0) = (1, 0, \ldots, 0)$ without loss of generality. Next, application of the method from the proof of the Straightening–Out–Theorem to the explicit differential equation $\dot{x} = G(x)$ yields a C^μ–diffeomorphism Ψ between neighborhoods of $0 \in \mathbb{R}^n$. Its inverse Ψ^{-1} transforms (4) into $\tilde{f}(x)\dot{x} = (1, 0, \ldots, 0)$, where $\tilde{f} = f \circ \Psi$. Hence, (4) and

$$\tilde{f}(x)\dot{x}_1 = 1, \qquad \dot{x}_2 = 0, \; \ldots, \dot{x}_n = 0 \tag{8}$$

are C^μ–conjugate near 0.

We now straighten out the set $\tilde{f}^{-1}(0)$: By means of the Implicit Function Theorem there is a mapping η defined near $0 \in \mathbb{R}^{n-1}$ such that $\Phi\colon x \mapsto (x_1 + \eta(x_2, \ldots, x_n), x_2, \ldots, x_n)$

is a local C^s–diffeomorphism and, for all x near $0 \in \mathbb{R}^n$, $\tilde{f}(\Phi(x)) = 0$ iff $x_1 = 0$. After setting $\hat{f}(x) = \sigma \tilde{f}(\Phi(x))$, Φ^{-1} transforms (8) into

$$\hat{f}(x)\dot{x} = \sigma G(0). \tag{9}$$

Exploiting properties of $\hat{f}$, one can show that there is a C^s–mapping φ defined near $0 \in \mathbb{R}^n$ that fulfills $D_1\varphi(0) > 0$ and $\varphi(x)^r D_1\varphi(x) = \hat{f}(x)$ for all x. Hence, the mapping $x \mapsto (\varphi(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n)$ is a local C^s–diffeomorphism and transforms (9) into the normal form (2). After all, (4) and (2) are C^s–conjugate near 0. □

The following Theorem, which is proved in [Rei97], relates the case $r = 1$ of the above Theorem to the results of [Rab89, Med91, RB97]:

2.2 Theorem: *x_0 is a standard singular point of (1) iff the hypotheses of Theorem 2.1 are met for $f(x) = \det A(x)$, $G(x) = (\operatorname{adj} A(x))g(x)$, $r = 1$ and some $s \in \mathbb{N} \cup \{\infty, \omega\}$. In this case, the values of (3) and (7) coincide.* □

The simplest example to illustrate standard singular points is the normal form (2) with $r = 1$, $n = 2$ and $\sigma = -1$. Here, the set of standard singular points, all being attracting, is the hypersurface defined by $x_1 = 0$, all other points are regular. (See Fig. 1(a) for illustration.) Further, for any $x_{0,2} \in \mathbb{R}$,

$$x_1(t) = \pm\sqrt{-2t} \quad \text{and} \quad x_2(t) = x_{0,2}$$

define exactly two solutions for $t < 0$. No other solutions exist, except translations of the above solutions in time, and hence, (1) and (2) from Section 1 also hold here.

Since the cases $r > 1$ and $n > 2$ of the normal form (2) can be completely analyzed by similar simple considerations, the above Theorems provide powerful tools to analyze the behaviour of the original differential equations (1) and (4) near points at which A is singular and f is zero, respectively.

3 An Example

While examples of differential equations with standard singular points have been given in the literature [Rab89, Rei96, RB97], we give a practical example for the case $r = 2$ below, which has resisted an analytic treatment up to now.

Byrne und Ho (see [BH87]) give the differential equations

$$0 = \pi\sqrt{\frac{R}{2\rho}}(R-y)^2\sqrt{-P'}\left(2.5\ln\left(\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\sqrt{-P'}-5\right)+10.5\right) - bQ_{c0} - \frac{P_0}{P}Q_{c0}(1-b), \tag{10}$$

$$0 = 2\pi\sqrt{\frac{R}{2\rho}}\sqrt{-P'}\Bigg((2.5Ry - 1.25y^2)\ln\left(\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\sqrt{-P'}-5\right) + 3Ry - 2.125y^2 - 13.6R\mu\sqrt{\frac{2}{\rho R}}\frac{1}{\sqrt{-P'}}\Bigg) - Q_a \tag{11}$$

as a model of a pipeline problem: To successfully pipe a certain foam, the foam is surrounded by an incompressible lubricating film. The question of interest is whether it is possible to keep the pressure sufficiently high in the whole pipeline.

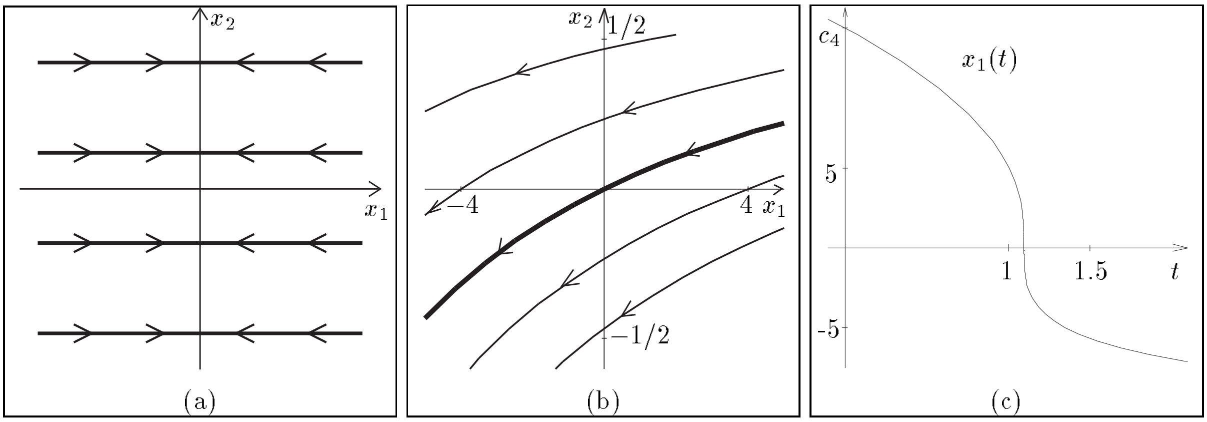

Figure 1: (a) Trajectories of the normal form (2) with $r = 1$, $n = 2$ and $\sigma = -1$. (b) x_1– and x_2–components of trajectories of (14)–(15). (c) x_1–component of those two solutions of (14)–(15) that correspond to the trajectories thickened in (a).

In (10)–(11), the (dropped) argument t of P, P' and y is the space coordinate along the pipeline in cm. y is the thickness of the lubricating film in cm, P is the pipeline pressure in $10^{-5}Ncm^{-2}$, and $P_0 = P(0) = 1.378 \cdot 10^8$ is the pressure at the beginning of the pipeline. R, ρ, μ, Q_a, Q_{c0}, and b are prescribed parameters: $R = 45.72$, $\rho = 0.814$, $\mu = 0.098$, $b = 0.345$, $Q_{c0} = 1.7153 \cdot 10^6$, and $Q_a = 3.027 \cdot 10^5$.

The equations (10)–(11) make sense if P, $-P'$ and the argument of the logarithm are positive.

Using the substitutions $x_1(t) = 10^{-7}\ P(10^7\ t)$, $x_2(t) = (-P'(10^7\ t))^{-1/2}$, and $x_3(t) = y(10^7\ t)\,(-P'(10^7\ t))^{1/2}\,R^{-1}$ from [RR94] as well as $x_4(t) = \ln\left(\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\,(-P'(10^7\ t))^{1/2} - 5\right)$, we arrive at the differential equation

$$x_2^2 x_1' = -1, \tag{12}$$

$$h(x) = 0, \tag{13}$$

where $h(x) = \begin{pmatrix} x_1(4.2 + x_4)(1 - x_2x_3)^2 - c_2x_2(bx_1 + c_4(1 - b)) \\ x_3\left((2 - x_2x_3)x_4 + 2.4 - 1.7x_2x_3\right) - c_3 \\ e^{x_4} - c_1x_3 + 5 \end{pmatrix}$, $c_1 = \frac{R}{\mu}\sqrt{\frac{\rho R}{2}} \approx 2012$, $c_2 = \frac{\rho Q_{c0}}{2.5\pi\mu R c_1} \approx 19.72$, $c_3 = \frac{10.88}{c_1} + \frac{0.4\rho Q_a}{\pi\mu R c_1} \approx 3.485$, and $c_4 = 10^{-7}\ P_0 = 13.78$.

Reducing the index of (12)–(13) by differentiating (13) as usual results in

$$x_2^2 x_1' = -1, \tag{14}$$

$$h'(x)\dot{x} = 0. \tag{15}$$

This differential equation is obviously of the type (1) with $g(x) = (-1, 0, 0, 0)$ and

$$A(x) = \begin{pmatrix} x_2^2 & 0 & 0 & 0 \\ \begin{matrix}(1-x_2\,x_3)^2 \\ (4.2+x_4)-c_2\,x_2\,b\end{matrix} & \begin{matrix}-2\,x_1\,x_3\,(1-x_2\,x_3)\,(4.2+x_4)- \\ c_2\,(c_4\,(1-b)+x_1\,b)\end{matrix} & \begin{matrix}-2\,x_1\,x_2\,(1-x_2\,x_3) \\ (4.2+x_4)\end{matrix} & x_1\,(1-x_2\,x_3)^2 \\ 0 & -x_3^2\,(1.7+x_4) & 2\begin{pmatrix}1.2-1.7\,x_2\,x_3 \\ +x_4-x_2\,x_3\,x_4\end{pmatrix} & x_3\,(2-x_2\,x_3) \\ 0 & 0 & -c_1 & e^{x_4} \end{pmatrix}.$$

We investigate (14)–(15) near the point $x_0 = (0, 0, x_{0,3}, x_{0,4})$, where $x_{0,3}$ is the solution of $c_3 = 2x_3(\ln(c_1x_3 - 5) + 1.2)$, i.e., $x_{0,3} \approx 0.237$, and $x_{0,4} = \ln(c_1x_{0,3} - 5) \approx 6.16$: Since (S_1) and (S_2) are fulfilled, (14)–(15) has exactly the same solutions near x_0 as (4) with $f(x) = \det A(x)$ and $G(x) = (\operatorname{adj} A(x))g(x)$ (Prop. 1.1). One can also show that the hypotheses of Theorem 2.1 are fulfilled with $\sigma = -1$ [Rei97]. Hence, the differential equation (14)–(15) is C^ω–conjugate near x_0 to the normal form (2) near 0 with $n = 4$, $r = 2$ and $\sigma = -1$.

The first two components of some trajectories of (14)–(15) are shown in Fig. 1(b). The first components of the solutions that correspond to the two thick trajectories in Fig. 1(b) are shown in Fig. 1(c). One can show that these solutions are indeed solutions of (12)–(13) and that only that solution that has positive x_1– and x_2–components makes sense for the practical problem.

One concludes from our analysis based on Theorem 2.1 that the pressure in the pipeline tends to 0 and that the flow chokes if the pipeline is sufficiently long. From the numerical simulation we see that the minimum length of the pipeline for this to happen is about $110km$. Further investigation shows that the conjecture from [BH87] that choking flow corresponds to a vanishing argument of the logarithms in (10)–(11) is incorrect.

4 Conclusions

It has been shown that implicit differential equations $A(x)\dot{x} = g(x)$ about certain singular points may be transformed by diffeomorphisms into the normal form $x_1^r\dot{x}_1 = \pm 1$, $\dot{x}_2 = \cdots = \dot{x}_n = 0$ about 0. This result is new and provides a deeper understanding of sytems' behavior near singular points. In particular, we have been able to analyze an example that had resisted an analytic treatment before. Further, the results of [Rab89, Med91] on standard singular points follow from ours.

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