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A normal form for implicit differential equations near singular points and its application

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This paper concerns quasi–linear implicit differential equations of form

$$\begin{aligned} 0 &= A_1(x)\dot{x} - g_1(x), \\ 0 &= g_2(x), \end{aligned} \tag{1}$$

where $A_1\colon U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n-m}) \in C^1$, $g_1\colon U \to \mathbb{R}^{n-m} \in C^1$, $g_2\colon U \to \mathbb{R}^m \in C^2$, $U \subseteq \mathbb{R}^n$ *is open,* $n, m \in \mathbb{N}$, *and* $m < n$. *In particular, (1) is considered about impasse points* $x_0 \in U$, *i.e., points* x_0 *beyond which solutions are not continuable. We review a recent result on a normal form about such points and discuss two examples.*

1 Introduction

Many technical systems and processes may be modelled by implicit differential equations of the form

$$A(x)\dot{x} = g(x), \tag{2}$$

where $A\colon U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \in C^1$ is a matrix of C^1–functions, $g\colon U \to \mathbb{R}^n \in C^1$, $U \subseteq \mathbb{R}^n$ is open, and $n \in \mathbb{N}$. Whenever A is regular at some point $x_0 \in U$, the above *implicit* differential equation (2) is locally equivalent to the *explicit* ordinary differential equation $\dot{x} = A(x)^{-1}g(x)$ in some neighborhood of x_0, and the usual existence, uniqueness and smoothness results apply.

If $A(x_0)$ is not regular, these results do not apply, and the analysis of (2) near x_0 becomes more difficult. For example, it may happen that the set of points at which A is singular forms an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$ containing x_0, a situation first studied by Rabier for $n > 1$ [1]. Rabier considered a special type of impasse points, i.e., points beyond which solutions of (2) are not continuable, and called them *standard singular points* [1]. More recently, Medved' [2], Reißig [3] and Reißig and Boche [4, 5] obtained normal forms for (2) near impasse points that are not necessarily standard.

If the matrix A from (2) is singular on some whole neighborhood of x_0, the above results do not apply, although the point x_0 may very well be an impasse. For example, Rabier and Rheinboldt [6] have investigated the quasi–linear differential equation (2) near special impasse points, called *standard impasse points* [6, p. 445]. Among other things, these points have the property that the corresponding point of the reduction of (2) by using local coordinates of of the hidden constraint set $\{x \in U \mid g(x) \in \operatorname{im} A(x)\}$ is a standard singular point of that reduction.

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VENKATASUBRAMANIAN, SCHÄTTLER and ZABORSKY [7] and REIßIG [8] have considered the semi–explicit case of (1). The results of [7] and [8] for standard impasse points are equivalent to those of [6] in the semi–explicit case, except that considerably weaker smoothness assumptions are made in [8] and nondifferentiable solutions are taken into account in [8]. In addition, both [7] and [8] also investigate more general types of impasse points.

WINKLER [9] has considered the special form (1) of (2) under the additional assumption that g_2 is a submersion and A_1 is surjective everywhere.

More recently, REIßIG and BOCHE [10] have shown the following for (1):

1.1 Theorem: *Consider the implicit differential equation (1), let $n, m, r \in \mathbb{N}$, $m < n$, $s \in \mathbb{N} \cup \{\infty, \omega\}$, and let $\mu = r + s - 1$ if $s \in \mathbb{N}$ and $\mu = s$ otherwise. Let further be $U \subseteq \mathbb{R}^n$ an open neighborhood of x_0, and let A_1, g_1, g_2 and Dg_2 be of class C^μ.*

Let f, G, and σ, be defined by

$$\begin{aligned} f(x) &= \det \begin{pmatrix} Dg_2(x) \\ A_1(x) \end{pmatrix}, \\ G(x) &= \left(\operatorname{adj} \begin{pmatrix} Dg_2(x) \\ A_1(x) \end{pmatrix} \right) \begin{pmatrix} 0 \\ g_1(x) \end{pmatrix}, \\ \sigma &= \operatorname{sign}(D^r f(x_0) G(x_0)^r), \end{aligned} \tag{3}$$

and assume the following: $f(x_0) = 0$, $g_2(x_0) = 0$, $\sigma \neq 0$, $f^{-1}(0) \cap g_2^{-1}(0)$ is an $(n-m-1)$–dimensional C^s–submanifold of $\mathbb{R}^n$, and $D^j f(x) G(x_0)^j = 0$ holds for all j with $1 \leq j < r$ and for all $x \in f^{-1}(0) \cap g_2^{-1}(0)$.

Then the differential equation (1) is C^s–conjugate near x_0 to the normal form

$$\begin{aligned} x_1^r \dot{x}_1 &= \sigma, \\ \dot{x}_2 = 0, \ \ldots, \ \dot{x}_{n-m} &= 0, \\ x_{n-m+1} = 0, \ \ldots, \ x_n &= 0 \end{aligned} \tag{4}$$

near 0. (For $n = m + 1$, the second row of (4) is missing.) If Σ is the C^s–diffeomorphism that transforms solutions of (4) into solutions of (1), then

$$(\lambda, 0, \ldots, 0) = D\Sigma(x_0) G(x_0) \tag{5}$$

holds for some $\lambda > 0$. □

In the above Theorem, C^ω is the class of analytic mappings, $D^r f$ is the r–th order derivative of f, and adj $A(x)$ denotes the transpose of the matrix of cofactors of $A(x)$. Further, a differential equation is C^r*–conjugate* near x_0 to another differential equation near $\tilde{x}_0$ if there are open neighborhoods U' and $\tilde{U}'$ of x_0 and $\tilde{x}_0$, respectively, and some C^r–diffeomorphism $\Phi \colon U' \to \tilde{U}'$ with $\Phi(x_0) = \tilde{x}_0$ such that a curve x with values in U' is a solution of the first equation iff the transformed curve $\Phi \circ x$ is a solution of the second.

It should be noted that the normal form (4) near 0 is C^ω–conjugate to the normal form

$$\begin{aligned} \dot{x}_1 &= 1, \\ \dot{x}_2 = \cdots = \dot{x}_{n-m} &= 0, \\ x_{n-m+1} = \cdots = x_{n-1} &= 0, \\ 0 &= x_1 - \sigma x_n^{r+1} \end{aligned} \tag{6}$$

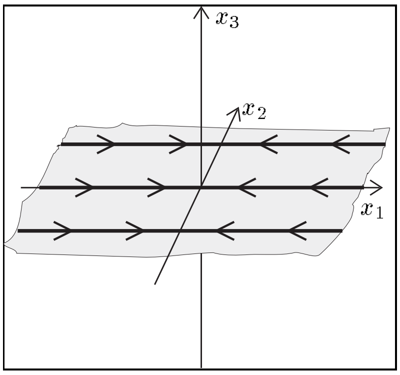

Figure 1: Trajectories of the normal form (4) with $r = 1$, $n = 3$, $m = 1$, and $\sigma = -1$.

near 0. (For $n = m + 1$, the second row of (6) is missing, and for $m = 1$, the third is missing.) That is, under the assumptions of the above Theorem, the implicit differential equation (1) near x_0 is C^s–conjugate to the normal form (6) near 0.

The following Corollary provides a sufficient condition for the hypotheses of Theorem 1.1 to be fulfilled:

1.2 Corollary: *The hypothesis of Theorem 1.1 that $f^{-1}(0) \cap g_2^{-1}(0)$ is an $(n - m - 1)$–dimensional C^s–submanifold of $\mathbb{R}^n$ is met if the remaining conditions of that Theorem hold and $f^{-1}(0)$ is an $(n - 1)$–dimensional C^s–submanifold of $\mathbb{R}^n$.* □

The following statement relates the case $r = 1$ of Theorem 1.1 to the results of [6, 7, 9, 8] on standard impasse points:

1.3 Corollary: *Let m, n, U, x_0 as in Theorem 1.1, let A_1 and g_1 be of class C^1, and let g_2 be of class C^2. In addition, assume that $A_1(x_0)$ is surjective.*
Then x_0 is a standard impasse point in the sense of [6] iff the requirements of Theorem 1.1 are fulfilled with $r = s = 1$. In that case, σ from (5) equals

$$\operatorname{sign}\left(\left\langle u \middle| \begin{pmatrix} 0 \\ g_1(x_0) \end{pmatrix} \right\rangle \cdot \left\langle u \middle| \begin{pmatrix} D^2 g_2(x_0)kk \\ A_1'(x_0)kk \end{pmatrix} \right\rangle\right)$$

for all $k \in \ker \begin{pmatrix} Dg_2(x_0) \\ A_1(x_0) \end{pmatrix} \setminus \{0\}$ and all $u \in \operatorname{im}\left(\begin{pmatrix} Dg_2(x_0) \\ A_1(x_0) \end{pmatrix}\right)^{\perp} \setminus \{0\}$, where $\langle \cdot | \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$. □

The normal form (4) with $r = 1$, $n = 3$, $m = 1$ and $\sigma = -1$, i.e.

$$\begin{aligned} x_1 \dot{x}_1 &= -1, \\ \dot{x}_2 &= 0, \\ x_3 &= 0 \end{aligned}$$

is a simple example to illustrate the phenomenon of impasse points. Here, all points of the x_2–axis are impasse points which are standard; see Fig. 1 for illustration. Further, for any $x_{2,0} \in \mathbb{R}$,

$$x_1(t) = \pm\sqrt{-2t}, \qquad x_2(t) = x_{2,0}, \qquad \text{and} \qquad x_3(t) = 0$$

define exactly two solutions for $t < 0$. No other solutions exist, except translations of the above solutions in time.

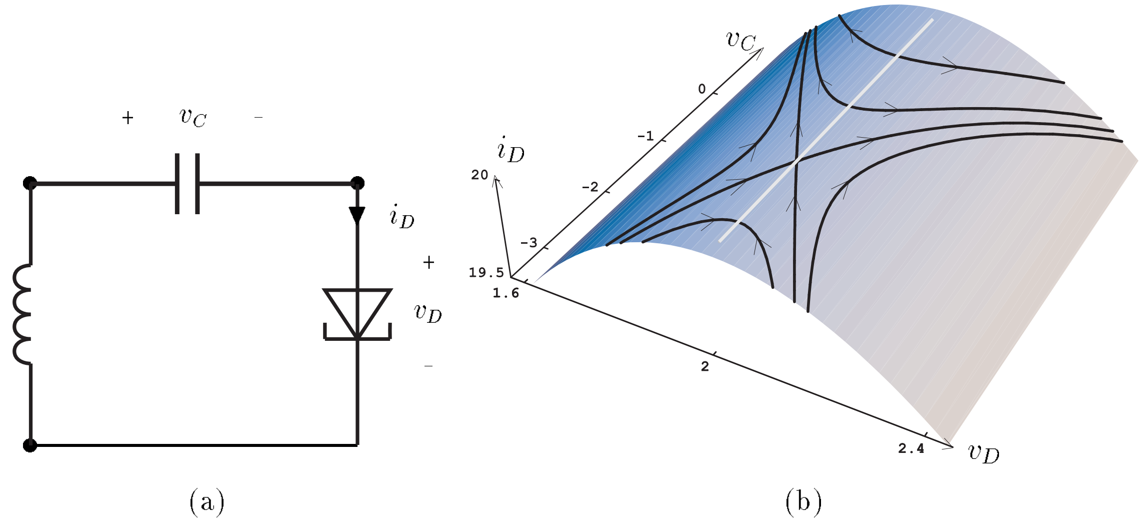

Figure 2: (a)Network investigated in Example 2.1. (b) The surface $g_2^{-1}(0)$, the set $f^{-1}(0) \cap g_2^{-1}(0)$ (grey line), and trajectories of (7)–(9).

Since the other cases of the normal forms (4) and (6) can be completely analyzed by similar simple considerations, the above Theorem provides a powerful tool to analyze the behaviour of the original differential equation (2) near certain impasse points.

This paper demonstrates the application of Theorems 1.1 to the analysis of two examples: An electrical network with impasse points from [8] and a pipeline problem with choking flow from [11].

2 Examples

2.1 Example: The network from [8, Example IV.4.], shown in Fig. 2(a), contains a capacitor of capacitance 1, an inductor of inductance 1, and a tunnel diode. It is described by the equations

$$\dot{v}_C = i_D, \tag{7}$$

$$\dot{i}_D = -v_C - v_D, \tag{8}$$

$$0 = i_D - v_D^3 + 9v_D^2 - 24v_D, \tag{9}$$

where (9) represents the voltage current relation of the tunnel diode.

Obviously, (7)–(9) is of form (1) with

$$A_1(v_D, v_C, i_D) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_1(v_D, v_C, i_D) = \begin{pmatrix} i_D \\ -v_C - v_D \end{pmatrix},$$

$$g_2(v_D, v_C, i_D) = i_D - v_D^3 + 9v_D^2 - 24v_D.$$

Hence,

$$f(v_D, v_C, i_D) = 3(6v_D - v_D^2 - 8),$$

$$G(v_D, v_C, i_D) = \begin{pmatrix} v_C + v_D \\ 3i_D(6v_D - v_D^2 - 8) \\ -3(v_C + v_D)(6v_D - v_D^2 - 8) \end{pmatrix}.$$

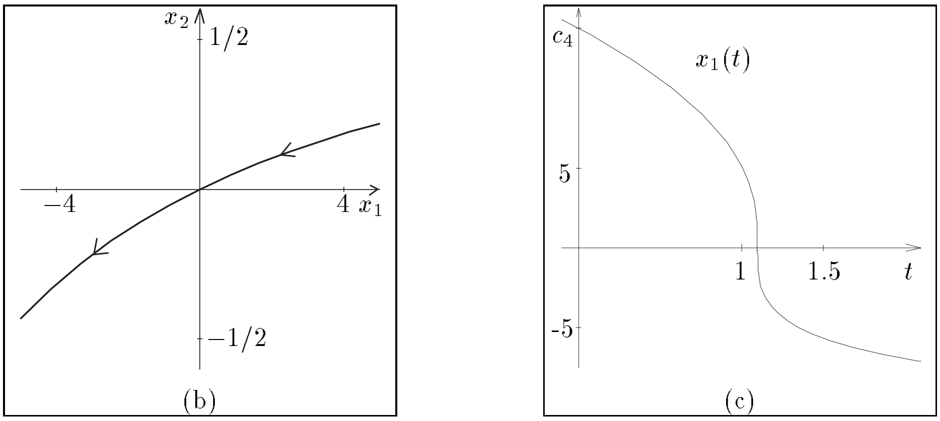

Figure 3: (b) x_1– and x_2–components of trajectories of (12)–(13). (c) x_1–component of the two solutions of (12)–(13) that correspond to the trajectories in (a).

Obviously, we have $f^{-1}(0) = \{2,4\} \times \mathbb{R}^2$ and $f^{-1}(0) \cap g_2^{-1}(0) = (\{2\} \times \mathbb{R} \times \{20\}) \cup (\{4\} \times \mathbb{R} \times \{16\})$. (See Fig. 2(b).)

For points$(v_D, v_C, i_D) \in f^{-1}(0)$ we have $f'(v_D, v_C, i_D)G(v_D, v_C, i_D) = 6(3 - v_D)(v_C + v_D)$, i.e.,

$$\sigma = ((3 - v_D)(v_C + v_D)).$$

Hence, the requirements of Theorem 1.1 are fulfilled with $r = 1$ and $s = \omega$ for all points $(v_D, v_C, i_D) \in f^{-1}(0) \cap g_2^{-1}(0)$ with $v_C + v_D \neq 0$. In particular, if $v_D = 2$, then $\sigma = -1$ for $v_C < -2$ and $\sigma = 1$ for $v_C > -2$. It follows from Theorem 1.1 that exactly two solutions converge at finite time to any point (v_D, v_C, i_D) with $v_D = 2$ and $v_C \neq -2$, where convergence is for increasing time if $v_C < -2$ and for decreasing time if $v_C > -2$. This is illustrated in Fig. 2(b).

From Corollary 1.3 we see that all points $(v_D, v_C, i_D) \in f^{-1}(0) \cap g_2^{-1}(0)$ except $(2, -2, 20)$ and $(4, -4, 16)$ are standard impasse points. $\square$

The following is a practical example for the case $r = 2$, which has resisted an analytic treatment up to now.

2.2 Example: Byrne und Ho (see [11]) give the differential equations

$$0 = \pi\sqrt{\frac{R}{2\rho}}(R-y)^2\sqrt{-P'}\left(2.5\ln\left(\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\sqrt{-P'} - 5\right) + 10.5\right) - bQ_{c0} - \frac{P_0}{P}Q_{c0}(1-b), \tag{10}$$

$$\begin{aligned} 0 = {} & 2\pi\sqrt{\frac{R}{2\rho}}\sqrt{-P'}\Bigg((2.5Ry - 1.25y^2)\ln\left(\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\sqrt{-P'} - 5\right) \\ & + 3Ry - 2.125y^2 - 13.6R\mu\sqrt{\frac{2}{\rho R}}\frac{1}{\sqrt{-P'}}\Bigg) - Q_a \end{aligned} \tag{11}$$

as a model of a pipeline problem: To successfully pipe a certain foam, the foam is surrounded by an incompressible lubricating film. The question of interest is whether it is possible to keep the pressure sufficiently high in the whole pipeline.

In (10)–(11), the (dropped) argument t of P, P' and y is the space coordinate along the pipeline in *cm*. y is the thickness of the lubricating film in *cm*, P is the pipeline pressure in $10^{-5}Ncm^{-2}$, and $P_0 = P(0) = 1.378 \cdot 10^8$ is the pressure at the beginning of the pipeline. R, ρ, μ, Q_a, Q_{c0}, and b are prescribed parameters: $R = 45.72$, $\rho = 0.814$, $\mu = 0.098$, $b = 0.345$, $Q_{c0} = 1.7153 \cdot 10^6$, and $Q_a = 3.027 \cdot 10^5$.

The equations (10)–(11) make sense if P, $-P'$ and the argument of the logarithm are positive.

By means of the substitutions $x_1(t) = 10^{-7}\ P(10^7\ t)$, $x_2(t) = \left(-P'(10^7\ t)\right)^{-1/2}$, $x_3(t) = y(10^7\ t)\left(-P'(10^7\ t)\right)^{1/2} R^{-1}$ from [12] as well as $x_4(t) = \ln\left(y\sqrt{\rho R/2}\left(-P'(10^7\ t)\right)^{1/2}/\mu - 5\right)$ we arrive at the differential equation

$$x_2^2 \dot{x}_1 = -1, \tag{12}$$

$$g_2(x) = 0, \tag{13}$$

where $g_2(x) = \begin{pmatrix} x_1(4.2 + x_4)(1 - x_2 x_3)^2 - c_2 x_2(b x_1 + c_4(1 - b)) \\ x_3\left((2 - x_2 x_3)x_4 + 2.4 - 1.7 x_2 x_3\right) - c_3 \\ e^{x_4} - c_1 x_3 + 5 \end{pmatrix}$, $c_1 = \frac{R}{\mu}\sqrt{\frac{\rho R}{2}} \approx 2012$, $c_2 = \frac{\rho Q_{c0}}{2.5 \pi \mu R c_1} \approx 19.72$, $c_3 = \frac{10.88}{c_1} + \frac{0.4 \rho Q_a}{\pi \mu R c_1} \approx 3.485$, and $c_4 = 10^{-7}\ P_0 = 13.78$.

(12)–(13) is obviously of form (1) if we set $g_1(x) = 1$ and $A_1(x) = (x_2^2, 0, 0, 0)$. We investigate (12)–(13) near the point $x_0 = (0, 0, x_{0,3}, x_{0,4})$, where $x_{0,3}$ is the solution of $c_3 = 2x_3(\ln(c_1 x_3 - 5) + 1.2)$, i.e., $x_{0,3} \approx 0.237$, and $x_{0,4} = \ln(c_1 x_{0,3} - 5) \approx 6.16$. To this end, we first set

$$A(x) = \begin{pmatrix} Dg_2(x) \\ A_1(x) \end{pmatrix} = \left(\begin{array}{c|ccc} \substack{(1 - x_2 x_3)^2 \cdot \\ (4.2 + x_4) - c_2 x_2 b} & & & \\ 0 & & \widetilde{A}(x) & \\ 0 & & & \\ \hline x_2^2 & 0 & 0 & 0 \end{array}\right),$$

where $\widetilde{A}(x)$ is defined as

$$\begin{pmatrix} \substack{-2 x_1 x_3 (1 - x_2 x_3) \cdot \\ (4.2 + x_4) - \\ c_2 (c_4 (1 - b) + x_1 b)} & \substack{-2 x_1 x_2 (1 - x_2 x_3) \cdot \\ (4.2 + x_4)} & x_1 (1 - x_2 x_3)^2 \\ -x_3^2 (1.7 + x_4) & 2 \begin{pmatrix} \substack{1.2 - 1.7 x_2 x_3 \\ + x_4 - x_2 x_3 x_4} \end{pmatrix} & x_3 (2 - x_2 x_3) \\ 0 & -c_1 & e^{x_4} \end{pmatrix}.$$

We obtain

$$\det \widetilde{A}(x_0) = \frac{(b - 1) c_1 c_2 c_4}{x_{0,3}} \left(2 x_{0,3}^2 + \frac{c_3}{c_1}(c_1 x_{0,3} - 5)\right) < 0.$$

Let f, G, and σ be as in Theorem 1.1. Obviously, $f(x_0) = 0$. Since $f(x) = -x_2^2 \det \widetilde{A}(x)$, $f(x) = 0$ is equivalent to $x_2 = 0$ in some neighborhood of x_0. In particular, $f^{-1}(0)$ is a C^ω–submanifold of $\mathbb{R}^4$ near x_0. Further, we have $f'(x)h = -2x_2 h_2 \det \widetilde{A}(x) - x_2^2 (\det \widetilde{A}(\cdot))'(x)h$ for all $h \in \mathbb{R}^4$, in particular, $f'(x) = 0$ if $x_2 = 0$, regardless of the value of $g_2(x)$. For the second order derivative of f we have $f''(x_0)G(x_0)^2 = -2(G(x_0)_2)^2 \det \widetilde{A}(x_0)$. Further, $\dim \ker A(x_0) = 1$ and $(0, 0, 0, g_1(x_0)) \notin \operatorname{im} A(x_0)$ imply $G(x_0) \in \ker A(x_0) \setminus \{0\}$. Since $\widetilde{A}(x_0)$ is regular, the lower–right 2×2–submatrix of $\widetilde{A}(x_0)$ is regular, and hence, $G(x_0)_2 \neq 0$. (Otherwise, we had $G(x_0)_3 = G(x_0)_4 = 0$. By $x_{0,4} + 4.2 \neq 0$, we also had $G(x_0)_1 = 0$, in contradiction to $G(x_0) \neq 0$.) Hence, hypotheses of Theorem 1.1 are fulfilled with $n = 4$, $m = 3$, $r = 2$, $\sigma = 1$, and $s = \omega$.

The first two components of two trajectories of (12)–(13) are shown in Fig. 3(b), and the first components of the corresponding solutions are shown in Fig. 3(c). Note that only that solution that has positive x_1– and x_2–components makes sense for the practical problem.

Finally, one shows that $G(x_0)_1 < 0$ and concludes from Theorem 1.1 that the pressure in the pipeline tends to 0 and that the flow chokes, provided that the pipeline is sufficiently long. From the numerical simulation we see that the minimum length of the pipeline for this to happen is about $110km$. In addition, further investigation shows that the conjecture from [11] that choking flow corresponds to a vanishing argument of the logarithms in (10)–(11) is incorrect. □

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