COMPUTING THE GENERIC INDEX OF THE CIRCUIT EQUATIONS OF LINEAR ACTIVE NETWORKS

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ABSTRACT

In many problems, such as the analysis of electrical circuits, one has to deal with Differential-Algebraic Equations (DAE's). Knowing their generic index is of interest for practical and numerical as well as for theoretical reasons. In this paper we give an algorithm that determines the generic index of the Circuit Equations of linear active electrical networks in polynomial time from the network topology alone. It uses integer arithmetic and does neither require symbolic calculations nor real arithmetic.

1 INTRODUCTION

In many problems one has to deal with Differential-Algebraic Equations (DAE's) of form

$$F(x(t), \dot{x}(t), t) = 0. \tag{1}$$

Knowing the *index* [1–3] of (1) would be a great advantage for several reasons. For example, numerical solution of (1) is, as a rule, the more complicated the higher its index is, and choosing an appropriate numerical algorithm is easier if the index is a priori known [1]. In particular, it would be advantageous if the index of DAE's to be solved in network analysis programs could be computed within these programs. Also, in case (1) describes the dynamic behavior of an electrical circuit, the index can be used to determine how many input derivatives enter the solution x, a question of high practical importance.

While computing the index for the general nonlinear case (1) is rather difficult and requires symbolic computations [2], its determination for the linear constant-coefficient case

$$A\dot{x}(t) + Bx(t) + q(t) = 0 \tag{2}$$

can be done using standard algorithms.

A serious problem, however, remains to be solved even in the linear case (2). If, for example, (2) are the Circuit Equations of a linear electrical network, then A and B contain the parameters of the network elements, such as resistances of resistors. Hence, the so computed index is correct for a special choice of parameters. In worst case it may happen that for *all* other values of the network's parameters the index differs from the computed one. This means that knowing the index of (2) for a special choice of parameters has limited practical importance only.

To overcome this, one has to compute a *generic index*, a number that equals the index of (2) for almost all values of the system's parameters.

A first result in that direction is due to DUFF and GEAR [4], a second due to REINSCHKE and RÖBENACK [5]. Unfortunately, these approaches have a drawback in common in that they require that any nonzero entry of the matrices A and B in (2) represents some parameter. In most applications, however, A and B contain both, parameters, e.g. resistances, and nonzero constants, e.g. 1 and -1 in KIRCHHOFF's Equations [6, Sec. 15]. Therefore, the algorithms suggested in [4, 5] do not apply, for example, to the Circuit Equations of electrical networks. Even worse, application of these algorithms to any system of equations coming from an electrical network in general requires symbolic calculations.

Another drawback of the algorithms mentioned above is that they are either not polynomial [4] or their complexity has not been considered yet [5].

There are other criteria on generic solvability of the Circuit Equations of linear active networks [7–10]. They can be used to determine whether the Circuit Equations are generically of index 1 and are checkable by operations on natural numbers. In addition, the results of [9, 7] characterize generically solvable networks and also yield the complexity of these networks.

In this paper we give an algorithm that determines the generic index of the Circuit Equations of linear active electrical networks in polynomial time from the network topology alone. It uses only integer arithmetic and is applicable to linear networks that consist of the following types of elements: Independent, possibly time-variant, voltage and current sources, time-invariant resistors, capacitors, inductors, nullator-norator pairs, and all four types of controlled sources.

After we will have explained basic terms in Section 2, the algorithm is stated in terms of matroids in Section 3. Although our algorithm is not intended to be performed by hand, we demonstrate its application to a simple circuit in Section 4. Additionally, our example shows that a linear network containing an OpAmp as the only active element can exhibit an arbitrarily high generic index.

2 PRELIMINARIES

As already mentioned, we consider linear networks that consist of the following types of elements: Independent, possibly time-variant, voltage and current sources, time-invariant resistors, capacitors, inductors, nullator-norator pairs, and controlled sources.

Consider such a network $\mathcal{N}$ with branch set $\{1, 2, \ldots, b\}$ and denote by $p \in \mathbb{R}^k$ the vector containing all parameters of $\mathcal{N}$: Resistances, capacitances, inductances, and gains of controlled sources. The Circuit Equations of $\mathcal{N}$ have the

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form

$$\underbrace{\begin{pmatrix} 0 \\ \hline 0 \\ \hline A_{LC}(p) \end{pmatrix}}_{A(p):=} \begin{pmatrix} v_1 \\ \vdots \\ v_b \\ i_1 \\ \vdots \\ i_b \end{pmatrix}^{\bullet} + \underbrace{\begin{pmatrix} K \\ \hline M(p) \\ \hline B_{LC} \end{pmatrix}}_{B(p):=} \begin{pmatrix} v_1 \\ \vdots \\ v_b \\ i_1 \\ \vdots \\ i_b \end{pmatrix} = q(t), \qquad (3)$$

where $(v_1, \ldots, v_b)$ and $(i_1, \ldots, i_b)$ denote the branch voltages and currents, respectively, K denotes a matrix corresponding to KIRCHHOFF's Equations, $M(p)$ is a matrix corresponding to the constitutive relations of the resistive branches, and the submatrices $A_{LC}(p)$ and B_{LC} correspond to the constitutive relations of the reactances.

We call the network $\mathcal{N}$ *generically solvable* if $(A(p), B(p))$ is a regular pencil [1, 3] for all p from some open dense set $U \subseteq \mathbb{R}^k$. Since we assume the functions of the independent sources, q, to be sufficiently smooth or to belong to some suitable function space, this terminology is completely justified [3, 11].

We say that $\mathcal{N}$ *is generically of index* μ if $\mathcal{N}$ is generically solvable and $\operatorname{ind}(A(p), B(p)) = \mu$ for all p from some open dense set $U \subseteq \mathbb{R}^k$, where $\operatorname{ind}(A(p), B(p))$ denotes the index of the matrix pencil $(A(p), B(p))$ [1, 3]. In case $\mathcal{N}$ is generically of index μ, we also write $\operatorname{ind}_g(\mathcal{N}) = \mu$.

3 THE ALGORITHM

In order to give a formulation of our algorithm that is both, precise and concise, we still need to introduce some further notation.

First, we denote the set of capacitor branches of a network $\mathcal{N}$ by C and the set of inductor branches by L.

If the branch set of the network $\mathcal{N}$ is $\{1, 2, \ldots, b\}$, it will be convenient to set

$$S := \{1, 2, \ldots, 2b\}$$
$$V_j := j$$
$$I_j := j + b$$

for all branches j of $\mathcal{N}$, i.e., for $1 \leq j \leq b$. The columns of the matrices $A(p)$ and $B(p)$ in (3) can then be denoted by V_1 through V_b and I_1 through I_b, a very natural notation.

In order to handle subsets of the column set of these matrices in the same way, we set

$$V_J := \{\, V_j \mid j \in J \} \quad \text{and} \quad I_J := \{\, I_j \mid j \in J \}$$

for any subset J of the branch set of $\mathcal{N}$.

A *matroid* [12] is a pair (G, F) of sets, G being its ground set and F being a set of subsets of G, called independent sets. An independent set of maximal cardinality is called *base*. The *dual matroid* of (G, F) is denoted by $(G, F)^*$, and the *sum* of the two matroids (G, F_1) and (G, F_2) is denoted by $(G, F_1) \vee (G, F_2)$ [12].

Let $Q(p)$ be a parameter dependent matrix that depends analytically on the parameter $p \in \mathbb{R}^k$. The *column matroid* $\mathcal{M}_C(Q)$ *of* Q is defined to be the pair (S, F), where

$$F := \{\, X \subseteq S \mid \exists_{p \in \mathbb{R}^k} \text{ The column set } X \text{ of } Q(p) \text{ is linearly independent} \}.$$

The above definition is correct, i.e., $\mathcal{M}_C(Q)$ is a matroid.

Table 1. Algorithm that determines the generic index of the Circuit Equations of a network.

Input	Matrices from (3) corresponding to the network $\mathcal{N}$.
Step 1	Find a common independent set $\mathcal{B}$ of maximal cardinality of $\mathcal{M}_C(K) \vee \mathcal{M}_C(M)$ and $\mathcal{M}_C(A_{LC} + B_{LC})^*$ such that $\lvert (I_C \cup V_L) \cap \mathcal{B} \rvert$ is maximal.
Step 2	If $\lvert\mathcal{B}\rvert < 2b - \lvert L \cup C \rvert$, then goto Sing.
Step 3	For $1 \leq i, j \leq 2b$ do the following: Set $\begin{pmatrix} K \\ M \\ A_{LC} + B_{LC} \end{pmatrix}_{i,j}$ to 1, and set all other elements of the ith line and jth column of that matrix to 0. Denote the resulting submatrices by $\widetilde{K}$, $\widetilde{M}$, and $\widetilde{A}_{LC} + \widetilde{B}_{LC}$. Find a common independent set $\widetilde{\mathcal{B}}$ of maximal cardinality of $\mathcal{M}_C(\widetilde{K}) \vee \mathcal{M}_C(\widetilde{M})$ and $\mathcal{M}_C(\widetilde{A}_{LC} + \widetilde{B}_{LC})^*$ such that $\lvert (I_C \cup V_L) \cap \widetilde{\mathcal{B}} \rvert$ is maximal. If $\lvert\widetilde{\mathcal{B}}\rvert < 2b - \lvert L \cup C \rvert$, then set $k_{i,j} = 0$. Otherwise, set $k_{i,j} = \lvert (I_C \cup V_L) \cap \widetilde{\mathcal{B}} \rvert$.
Out	Print("The network is generically of index ", $1 + \max_{1 \leq i,j \leq 2b} k_{i,j} - \lvert (I_C \cup V_L) \cap \mathcal{B} \rvert$); End.
Sing	Print("The network is not solvable."); End.

Using this notation and denoting the cardinality of any set H by $\lvert H \rvert$, the algorithm is given in Table 1.

The following Theorems imply the correctness of the algorithm. Unfortunately, their proofs are too lengthy to be conveniently given here. (Although formulated differently, Theorems 3.1. and 3.3. have been known earlier [7–9].)

3.1. Theorem: *The following statements are equivalent.*

(i) $\mathcal{N}$ *is generically solvable.*

(ii) There is a common independent set of $\mathcal{M}_C(K) \vee \mathcal{M}_C(M)$ *and* $\mathcal{M}_C(A_{LC} + B_{LC})^*$ *of cardinality* $2b - \lvert C \cup L \rvert$. □

3.2. Theorem: *Let* $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ *be* $n \times n$ *matrices,* $n \in \mathbb{N}$*, such that* (A, B) *is a regular pencil. Then*

$$1 + \max_{1 \leq i,j \leq n} (\deg_s(\operatorname{adj}(sA + B))_{i,j}) - \deg_s \det(sA + B)$$

is the index of (A, B)*, where* $\operatorname{adj}(sA + B)$ *is the transpose of the matrix of cofactors of* $sA + B$ *and* $\deg_s$ *denotes the degree of a polynomial in* s. □

3.3. Theorem: *Let* $\mathcal{B}$ *be a common independent set of* $\mathcal{M}_C(K) \vee \mathcal{M}_C(M)$ *and* $\mathcal{M}_C(A_{LC} + B_{LC})^*$ *of cardinality* $2b - \lvert C \cup L \rvert$ *such that* $\lvert (I_C \cup V_L) \cap \mathcal{B} \rvert$ *is maximal. Then the degree of the polynomial* $s \mapsto \det(sA(p) + B(p))$ *equals* $\lvert (I_C \cup V_L) \cap \mathcal{B} \rvert$ *generically.* □

3.4. Theorem: *Let* $1 \leq i, j \leq 2b$ *and let* $\widetilde{A}_{LC} + \widetilde{B}_{LC}$*,* $\widetilde{K}$*,* $\widetilde{M}$*, and* $k_{i,j}$ *be as constructed in Step 3 of the algorithm* *given in Table 1. Then $k_{i,j}$ equals*

$$\deg_s\left(\operatorname{adj}\begin{pmatrix}\widetilde{K}\\ \widetilde{M}(p)\\ s\widetilde{A}_{LC}(p)+\widetilde{B}_{LC}(p)\end{pmatrix}\right)_{j,i}$$

generically. □

To see that the algorithm in Table 1 requires integer arithmetic only and that the number of operations to be performed is bounded by a polynomial in the number b of branches of the network, we remark the following.

We can determine the generic rank of any submatrix of K, $\widetilde{K}$, M, $\widetilde{M}$, $A_{LC}+B_{LC}$, and $\widetilde{A}_{LC}+\widetilde{B}_{LC}$ by using integer arithmetic. In case of K and $\widetilde{K}$, this is obvious. For the remaining matrices, calculating the *term rank* [12] suffices. In any case, the number of operations performed is bounded by a polynomial in b.

Finding the sets $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ in Steps 1 and 3 of the algorithm is now simple by *matroid partition* and *intersection* algorithms [12].

Let us finally remark that the proposed algorithm can be easily adapted to compute the number of input derivatives that enter all or a certain part of network variables.

4 APPLICATION OF THE ALGORITHM TO A SIMPLE CIRCUIT

Consider the network structure given in Fig. 1(a) which consists of l branches. One can show that it is of generic index $l-1$, i.e., its generic index equals the number of reactances plus 1. In order to demonstrate how the algorithm given in the previous section works, we will apply it to the network shown in Fig. 1(b), a special case of the structure from Fig. 1(a).

The matrices $A(p)$ and $B(p)$ of the Circuit Equations of the network from Fig. 1(b) are

$$A(p)=\left(\begin{array}{cccccccccc}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&L_3&\cdot&\cdot\\
\cdot&\cdot&\cdot&C_4&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&L_5
\end{array}\right)$$

and

$$B(p)=\left(\begin{array}{cccccccccc}
1&-1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&-1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&-1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot\\
\hline
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\hline
\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot\\
\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right),$$

where dots denote zeros. Symbolic calculations yield that

$$\det(sA(p)+B(p))=1$$

and that the generic degrees of the elements of $\operatorname{adj}(sA(p)+B(p))$ are

$$\begin{pmatrix}
\cdot&2&\cdot&1&3&3&2&2&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&2&\cdot&1&3&3&2&2&1&\cdot\\
\cdot&1&\cdot&\cdot&2&2&1&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&2&2&1&1&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&2&2&1&1&\cdot&\cdot
\end{pmatrix}.$$

Hence, the generic index of the network is $4=1+3-0$.

We now calculate this result with the help of our algorithm from Section 3. In Step 1, matroid partition and intersection algorithms yield the common independent set

$$\mathcal{B}=\{V_1,V_2,V_4,I_1,I_2,I_3,I_5\}.$$

(Performing these matroid algorithms by hand is rather tedious. We therefore emphasize again that our algorithm is not intended to be performed by hand.)

The network is generically solvable since $|\mathcal{B}|=7=10-3$ (Step 2), i.e., $\mathcal{B}$ is even a common base. Further,

$$\deg_s\det(sA(p)+B(p))=|(I_C\cup V_L)\cap\mathcal{B}|=0$$

holds generically.

Let us calculate $\widetilde{\mathcal{B}}$ in Step 3 for the special case $i=j=6$. $\mathcal{M}_C(\widetilde{K})\vee\mathcal{M}_C(\widetilde{M})$ and $\mathcal{M}_C(\widetilde{A}_{LC}+\widetilde{B}_{LC})$ have exactly the two bases

$$\{V_1,V_2,V_3,I_1,I_2,I_4,I_5\}\quad\text{and}\quad\{V_1,V_2,V_4,I_1,I_2,I_3,I_5\}$$

in common. The first yields maximal $|(I_C\cup V_L)\cap\widetilde{\mathcal{B}}|=2$. (Note that it is not necessary to know all common bases; the matroid partition and intersection algorithms automatically give the maximal - the first here - common base.) Hence, $k_{6,6}=2$.

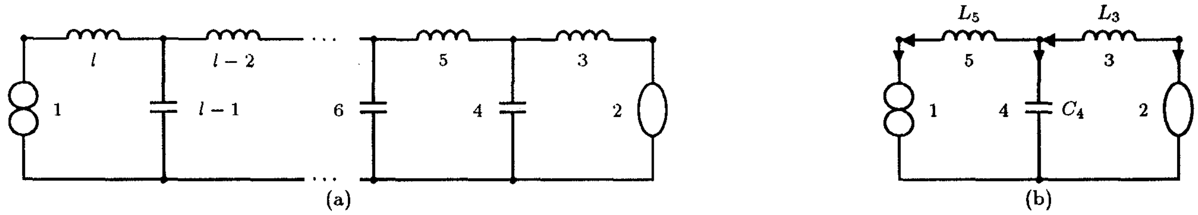

Figure 1. (a) Simple network of generic index $l-1$. (b) Network $\mathcal{N}$ to which the algorithm is applied.

5 CONCLUSIONS

An algorithm that computes the generic index of the Circuit Equations of linear active networks has been given. It uses integer arithmetic and does neither require symbolic calculations nor real arithmetic. The number of operations to be performed is bounded by a polynomial in the number of branches of the network in question. The algorithm is, however, by no means optimal; to make it faster is a topic of future research.

In addition to the calculation of the generic index of the Circuit Equations of networks, the algorithm may also be used as a tool for proofs of theorems concerning the index. It can also be easily adapted to compute the number of input derivatives that enter all or a certain part of network variables.

The algorithm has been formulated in terms of matroids; an equivalent formulation in terms of conjugate trees [13] is currently in preparation. Research should also be devoted to a generalization to more general parameter dependent matrix pencils - including those corresponding to networks that contain ideal transformers - and to nonlinear DAE's.

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