On Optimal Network Regularization

Klaus Röbenack and Gunther Reißig

Abstract

In this paper we define classes of impasse points, an important phenomenon of Differential-Algebraic Equations (DAEs). Focussing on DAEs occuring in the analysis of electrical networks, a brief discussion of properties of these classes and examples are given. Some of these examples are counterexamples to published assertions on impasse points.

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ON OPTIMAL NETWORK REGULARIZATION

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Abstract **A new method for an index one regularization of networks using a minimal number of additional elements is presented. It is applicable to linear electrical networks that may contain independent voltage and current sources, time–invariant resistors, capacitors, inductors, and nullator–norator pairs. The algorithm uses the network topology alone and is based on the concept of normal pairs of conjugate trees.**

I. INTRODUCTION

Many real-world systems, especially electrical networks, can be modelled by differential algebraic equations (DAE). The *index* of a DAE plays a crucial role concerning analytical and numerical properties [1–3]. On higher index problems (index greater than one) small pertubations of the input may cause arbitrary large errors in the solution [2].

One approach to overcome the disadvantages related to higher index problems in circuit simulation is the modification of the electrical network. (A similar approach, the so–called feedback-regularization of control systems has been discussed in literature [4].) In this context the following two problems may be formulated:

Problem 1 Regularization *by augmenting the network with additional elements such that the modified network is generically of index one.*

Problem 2 (Optimal Network Regularization) *Solution of Problem 1 using a minimal number of additional network elements.*

In this paper, we consider the following modifications to be admissible:

(i) Augmenting the network with either a resistor, a capacitor or an inductor in series to an existing branch.

(ii) Augmenting the network with either a resistor, a capacitor or an inductor as a new branch between two existing nodes.

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The first problem has been solved early for RLC networks [5,6]. A recent result for active networks can be found in [7, Cor. 4.10]. In this work we give an algorithm to solve the Problem 2 for a class of linear active electrical networks. Our algorithm uses the topology of the network only and is mainly based on the concept of normal pairs of conjugate trees. Further, the number of operations to be performed is bounded by a polynomial in the number of branches of the network.

II. PRELIMINARIES

We confine ourselves to linear networks containing the following types of elements: independent voltage and current sources, time–invariant resistors, capacitors, inductors, and nullator–norator pairs. Furthermore, we assume the network graph to be connected.

Now, we consider such a network $\mathcal{N}$ consisting of b branches and denote by $p \in \mathbb{R}^k$ the vector containing all parameters of $\mathcal{N}$, i.e., resistances, capacitances, and inductances. The *circuit equations* of $\mathcal{N}$ are given by

$$A(p)\,\dot{x} + B(p)\,x = q(t),$$

where A and B are parameter dependent $2b \times 2b$-matrices, and x denotes the $2b$-dimensional vector of the branch voltages and the branch currents.

A network $\mathcal{N}$ is said to be *generically solvable* if the associated matrix pencil $(A(p), B(p))$ is regular [8,1] for all p belonging to an open and dense subset of $\mathbb{R}^k$. A generically solvable network $\mathcal{N}$ is said to be generically of index ν if $\mathrm{ind}\,(\mathrm{A(p)}, \mathrm{B(p)}) = \nu$ holds for all p from an open and dense subset of $\mathbb{R}^k$, where $\mathrm{ind}\,(\mathrm{A(p)}, \mathrm{B(p)})$ denotes the index of the pencil $(A(p), B(p))$ [1,9].

III. THE ALGORITHM

In order to formulate our algorithm in terms of the network topology, we need to introduce some further notation, cf. [10–13].

Definition 1 *Let $\mathcal{G}$ be the undirected network graph of some network $\mathcal{N}$. A pair (t_1, t_2) of spanning trees*[†] *of $\mathcal{G}$ is called a* pair of conjugate trees (PCT) *if*

[†] We use the terms *tree* and *forest* [14] synonymously.

Table 1: Algorithm that yields a generically index one network using a minimal number of additional network elements

Input	Network $\mathcal{N}$.
Step 1:	Supplement $\mathcal{N}$ by additional inductor branches between each pair of different nodes, which are not yet connected by one branch. We denote the resulting network by $\mathcal{N}'$.
Step 2:	Determine a normal pair of conjugate trees (t_1, t_2) of $\mathcal{N}'$.
Step 3:	Construct a network $\mathcal{N}''$ that is obtained from $\mathcal{N}$ as follows: (i) Augment $\mathcal{N}$ with either a resistor or an inductor in series to each capacitor branch of $\mathcal{N}$ not contained in t_1. (ii) Augment $\mathcal{N}$ with either a resistor or a capacitor in parallel to each inductor branch of $\mathcal{N}$ contained in t_1. (iii) Augment $\mathcal{N}$ with either a resistor or a capacitor between each pair of nodes connected by an inductor branch that is contained in t_1 and $\mathcal{N}'$ but not in $\mathcal{N}$.
Output	Network $\mathcal{N}''$.

(i) t_1 *containes all norator branches, no nullator branch, and some resistor, capacitor or inductor branches, all voltage source branches, no current source branches,*

(ii) t_2 *contains all nullator branches, no norator branch, the same resistor, capacitor or inductor branches as* t_1*, and all voltage source branches, no current source branches.*

Let C and L denote the sets of capacitor and inductor branches of $\mathcal{N}$, respectively, and let us denote the cardinality of any set H by $|H|$.

Definition 2 *A pair of conjugate trees* (t_1, t_2) *is called* normal *if it maximizes the sum of the number of tree capacitors and the number of co-tree inductors among all pairs of conjugate trees of the network* $\mathcal{N}$ *in question, i.e., if*

$$\max\{|w_1 \cap C| + |L \setminus w_1| : (w_1, w_2) \text{ is a PCT of } \mathcal{N}\}$$

equals $|t_1 \cap C| + |L \setminus t_1|$.

The algorithm is given in Table 1. Its correctness is implied by the following Theorem:

Theorem 1 *Let* $\mathcal{N}$ *be a network containing independent sources, resistors, capacitors, inductors and nullator-norator pairs only. Furthermore, we assume that* $\mathcal{N}$ *is connected and has a pair of conjugate trees. Then the network* $\mathcal{N}''$ *obtained from the algorithm mentioned above solves Problem 2.*

Proof. Let us denote the set of inductor branches of $\mathcal{N}'$ by L', which implies $L \subseteq L'$. We assume that $\mathcal{N}$ has a PCT. The same branches constitute a PCT of $\mathcal{N}'$. Hence, $\mathcal{N}'$ contains a normal PCT (t_1, t_2).

First, we show that $\mathcal{N}''$ is generically of index one. For this purpose we consider the different modifications:

(a) Augmenting $\mathcal{N}'$ with an additional branch in series to each $c \in C \setminus t_1$ yields a new network $\mathcal{N}'_a$ with a PCT $(t_1 \cup C, t_2 \cup C)$.

(b) By augmenting $\mathcal{N}'_a$ with a branch in parallel to each $l \in L \cap t_1$ and denoting the set of supplemented branches by Z, we obtain a network $\mathcal{N}'_b$ with a PCT $((t_1 \cup C \cup Z) \setminus L, (t_2 \cup C \cup Z) \setminus L)$.

(c) Replacing each $l \in t_1 \cap (L' \setminus L)$ by either a resistor or a capacitor yields a network $\mathcal{N}'_c$ with the same PCT as $\mathcal{N}'_b$.

(d) Deleting each $l \in (L' \setminus L) \setminus t_1$ yields a network $\mathcal{N}'_d$ with the same PCT as $\mathcal{N}'_c$.

Now, if we choose the new branches in (a),(b) according to step 3 (i),(ii) of our algorithm, we have $\mathcal{N}'' = \mathcal{N}'_d$. The normal PCT of $\mathcal{N}''$ contains no inductor branch and all capacitor branches on $\mathcal{N}''$. Hence, the network $\mathcal{N}''$ is generically of index one [7, Cor. 4.9].

In the next part we consider the generic complexity σ of $\mathcal{N}'$, i.e., the degree of $s \mapsto \det(sA'(p) + B'(p))$ for all p from an open and dense subset of $\mathbb{R}^k$, where A' and B' are the matrices of the circuit equations of $\mathcal{N}'$. Let b' denote the number of branches of $\mathcal{N}''$, and let $\Delta b'' := b'' - b$. We have (cf. [7, Th. 4.5])

$$\begin{aligned} \sigma &= |C \cap t_1| + |L' \setminus t_1| \\ &= |C| - |C \setminus t_1| + |L'| - |L' \cap t_1| \\ &= |C| + |L'| - \underbrace{|C \setminus t_1|}_{(a)^{\ddagger}} - \underbrace{|L \cap t_1|}_{(b)^{\ddagger}} - \underbrace{|(L' \setminus L) \cap t_1|}_{(c)^{\ddagger}} \\ &= |C| + |L'| - \Delta b''. \end{aligned} \tag{1}$$

In order to show that $\mathcal{N}''$ contains a minimal number of additional network elements we consider a network $\tilde{\mathcal{N}}$ consisting of $\tilde{b}$ branches and obtained from $\mathcal{N}$ using the admissible modifications with resistor branches

‡Resulting from additional elements according to step 3(i)-(iii) in the algorithm

only (the other replacements can be treated similarly). Assume $\tilde{\mathcal{N}}$ to be generically of index one and minimal. Then, it can be verified that $\tilde{\mathcal{N}}$ may have been obtained from $\mathcal{N}$ using the following modifications only:

(M1) Augmenting $\mathcal{N}$ with a resistor in series to a capacitor.

(M2) Augmenting $\mathcal{N}$ with a resistor in parallel to an inductor.

(M3) Augmenting $\mathcal{N}$ with a resistor as a new branch between two nodes which are not yet connected by one branch of $\mathcal{N}$.

Let R_C, R_L and $R_{L'}$ denote the sets of additional resistor branches that result from applying (M1), (M2) and (M3), respectively. We denote a normal PCT of $\tilde{\mathcal{N}}$ by $(\tilde{t}_1, \tilde{t}_2)$ and the number of additional branches by $\Delta\tilde{b} := \tilde{b} - b = |R_C| + |R_L| + |R_{L'}|$. Because we assumed a generic index one we have $C \subseteq \tilde{t}_1$ and $L \cap \tilde{t}_1 = \emptyset$ [7, Cor. 4.9]. The generic complexity of $\tilde{\mathcal{N}}$ is given by $|C| + |L|$. Now, we modify $\tilde{\mathcal{N}}$ as follows:

(a) Contraction of each $r \in R_C$. The new network has a generic complexity not less than $|C| + |L| - |R_C|$.

(b) Delete each $r \in R_L$. The generic complexity will be not less than $|C| + |L| - |R_C| - |R_L|$.

(c) Replace each $r \in R_{L'}$ by an inductor. The lower bound of the generic complexity remains the same as in (b).

(d) Supplement the network by additional inductor branches between each pair of nodes not connected by one branch of $\tilde{\mathcal{N}}$. The generic complexity $\tilde{\sigma}$ of the resulting network fulfills

$$\begin{aligned} \tilde{\sigma} &\geq |C| + |L| - |R_C| - |R_L| + |L' \setminus L| - |R_{L'}| \\ &= |C| + |L'| - \Delta\tilde{b}. \end{aligned} \tag{2}$$

The network obtained here is nothing else than $\mathcal{N}'$, i.e., $\tilde{\sigma} = \sigma$. Because of Eqs. (1) and (2) we have $\Delta b'' \leq \Delta\tilde{b}$, i.e., the network $\mathcal{N}''$ has a minimal number of additional elements. ■

Step 1 as well as step 3 of the algorithm require at most b^2 simple network modifications. The normal pair of conjugate trees (step (ii)) can be determined using the method proposed in [15]. This implies that the number of operations of the whole algorithm is bounded by a polynomial in b.

Corollary 1 (To the proof of Theorem 1) *Let $\mathcal{N}$, $\mathcal{N}'$, C, L' and σ be defined as above. Then the minimal number of additional network elements to regularize $\mathcal{N}$ using the admissible modifications is equal to $|C| + |L'| - \sigma$.*

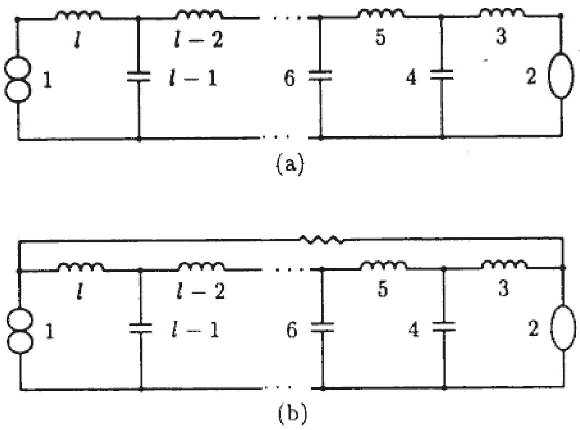

Figure 2: (a) Network $\mathcal{N}$ generically of index $l-1$, (b) Regularized Network $\mathcal{N}''$

IV. APPLICATIONS OF THE ALGORITHM

Example 1 Consider the network $\mathcal{N}$ given by Fig. 1(a). It is the linear counterpart of a so-called inverse system used to decode chaotically modulated signals [16]. This network is generically of index four. To obtain the associated network $\mathcal{N}'$ we have to augment $\mathcal{N}$ by a single inductor only. The normal pair of conjugate trees of $\mathcal{N}'$ has been drawn in Fig. 1(b),(c) using solid lines. In order to construct the network $\mathcal{N}''$ we have to augment $\mathcal{N}'$ with a first resistor or inductor in series to the capacitor not in t_1, and to replace the additional inductor (contained in t_1) by a second resistor or a capacitor. Because the normal pair of conjugate trees contains all capacitor branches and no inductor branch (see Fig. 1(d),(e)), the network is generically of index one. Note that the procedure proposed in [7] would require three instead of two additional elements as shown in Fig. 1(f).

Example 2 The network $\mathcal{N}$ given in Fig. 2(a) is generically of index $l-1$ [17]. Applying the usual procedure to regularize $\mathcal{N}$, one would require $l-2$ additional elements. Using the algorithm presented here we have to augment $\mathcal{N}$ by a single resistor only (Fig. 2(b)).

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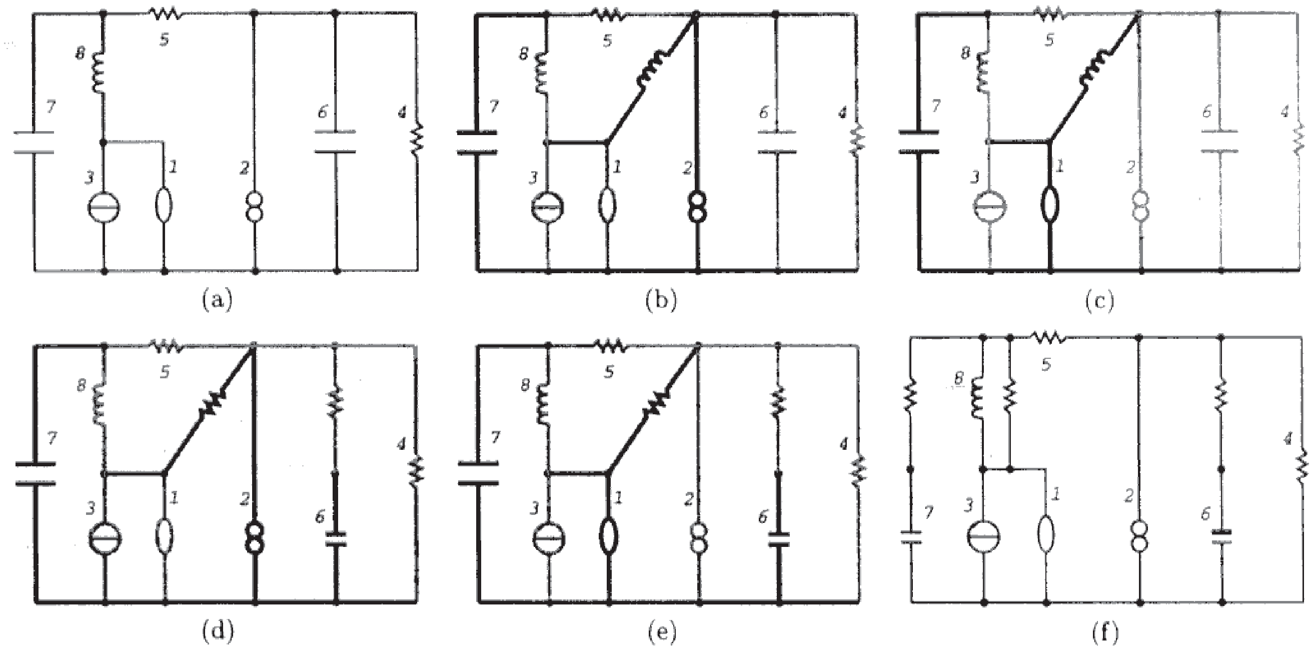

Figure 1: (a) Network $\mathcal{N}$, (b),(c) Network $\mathcal{N}'$ with a normal pair of conjugate trees, (d),(e) Resulting network $\mathcal{N}''$ with a normal pair of conjugate trees, (f) Network modified as in [7] without optimization

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