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# Structure at Infinity of Structure Matrix Pencils — A Toeplitz Matrix Approach

Gunter Wiedemann, Kurt J. Reinschke, and Gunther Reißig
Institut für Regelungs- und Steuerungstheorie
Technische Universität Dresden
01062 Dresden, Germany

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## Abstract

The structure at infinity of a matrix pencil can be obtained by rank determination of Toeplitz matrices. We show that the generic rank of these matrices equals the structural rank. Thus the Toeplitz matrix approach is also suited for investigations of structure matrix pencils. Computational results underline the efficiency of this approach.

## Introduction

The structure at infinity of matrix pencils $(sE-A)$, where $E, A \in \mathbb{R}^{p\times q}$, plays an important role in control theory. It provides information about disturbance decoupling [2], input/output decoupling [3], as well as difficulties to be expected if you want to solve the system $E\dot{x} = Ax$ numerically.

In this paper we deal with structure matrix pencils $[sE - A]$. For this purpose, only the "structure" of the matrices $E$ and $A$ is taken into account by mapping them into binary matrices $[E]$ and $[A]$.

Known approaches analyzing the structure at infinity [5, 7] essentially exploit the fact that the structure is related to the maximum degrees of some minors of $[sE-A]$. We propose another method: The structure at infinity of matrix pencils can also be determined by means of the rank of Toeplitz matrices. We show that the generic rank of these matrices equals the structural rank. Thus the Toeplitz matrix approach can be adapted to structure matrix pencils. A related result can be found in [8].

Using bipartite-matching algorithms for structural rank determination we obtain an algorithm for determining the structure at infinity of structure matrix pencils, which has the same complexity as the used matching algorithm.

The paper is organized as follows: First we provide some notions and preliminaries concerning the structure at infinity of matrix pencils $(sE - A)$. Then we deal with structure matrix pencils $[sE-A]$ and state our main result: the equality of the generic and the structural rank of a certain type of Toeplitz matrices. Finally we report the computational results.

## Preliminaries

Consider a (singular) matrix pencil $(sE - A)$, where $E, A \in \mathbb{R}^{p\times q}$. Its *normal rank* is defined by $\rho := \max_{s\in\mathbb{C}}(sE - A)$. Every matrix pencil can be brought into the *Kronecker Canonical Form (KCF)* (cf. [4]) by premultiplication with a nonsingular matrix $P \in \mathbb{R}^{p\times p}$ and postmultiplication with a nonsingular matrix $Q \in \mathbb{R}^{q\times q}$:

$$(PEQ, PAQ) = \left( \begin{pmatrix} I_{n_1} & & \\ & N & \\ & & E_{\text{sing}} \end{pmatrix}, \begin{pmatrix} J & & \\ & I_{n_2} & \\ & & A_{\text{sing}} \end{pmatrix} \right)$$

$J \in \mathbb{R}^{n_1\times n_1}$ is a Jordan matrix, $N \in \mathbb{R}^{n_2\times n_2}$ a nilpotent Jordan matrix. $E_{\text{sing}}$ and $A_{\text{sing}}$ contain the singular part of $(sE-A)$. Its structure is not needed in this paper. The structure of $N$, i. e. the sizes $\delta_1, \delta_2, \dots$ ($\delta_i \le \delta_j$ for $i < j$) of the 1st, 2nd, ... Jordan block, is called the *structure at infinity* of $(sE - A)$. Counting all Jordan blocks of the same size we obtain a list

$$\nu(E, A) := \{\nu_1, \nu_2, \dots, \nu_{n_\nu}\} \tag{1}$$

where $\nu_i = \text{card}\{\delta_j | \delta_j = i\}$ . The integer $n_\nu$ represents the size of the largest Jordan block of $N$, and it is called the *index* of $(sE - A)$.

The integers $\delta_\lambda$ can be obtained by determining the maximum degree of all minors of size $\text{rk}\, E + \lambda$ as well as the maximum degree of all minors of size $\text{rk}\, E + \lambda - 1$ (cf. [7]):

$$\begin{aligned} \delta_\lambda = 1 - &\max_{\substack{1\le i_1<\dots<i_{\text{rk}\,E+\lambda}\le p\\ 1\le j_1<\dots<j_{\text{rk}\,E+\lambda}\le q}} \deg(sE - A)^{i_1,i_2,\dots,i_{\text{rk}\,E+\lambda}}_{j_1,j_2,\dots,j_{\text{rk}\,E+\lambda}} \\ + &\max_{\substack{1\le i_1<\dots<i_{\text{rk}\,E+\lambda-1}\le p\\ 1\le j_1<\dots<j_{\text{rk}\,E+\lambda-1}\le q}} \deg(sE - A)^{i_1,i_2,\dots,i_{\text{rk}\,E+\lambda-1}}_{j_1,j_2,\dots,j_{\text{rk}\,E+\lambda-1}} \end{aligned} \tag{2}$$

Now consider a block Toeplitz matrix

$$M_\lambda(E,A) := \underbrace{\begin{pmatrix} E & & & & \\ A & E & & & \\ & A & E & & \\ & & \ddots & \ddots & \\ & & & A & E \end{pmatrix}}_{\lambda\cdot p\times\lambda\cdot q}.$$

The following Lemma 1 is an immediate consequence of [6], Corollary 1: The structure at infinity of $(sE - A)$ can be obtained by rank determination of $M_1(E,A), M_2(E,A), \ldots, M_{n_\nu+1}(E,A)$.

**Lemma 1**

$$\forall\lambda\in\mathbb{N}: \rho{-}\mathrm{rk}\, M_\lambda(E,A) + \mathrm{rk}\, M_{\lambda-1}(E,A) = \sum_{i=\lambda}^{n_\nu} \nu_i \quad (3)$$

We now state a lemma, which turns out to be the key for our main result in the next section. A product of matrix elements, each of which occupying different columns and rows, is a *term* of a minor of the matrix under consideration. By a *term* $U$ (or $T$) of $(sE - A)$ (or $M_\lambda(E,A)$) we mean a term of a minor of $(sE - A)$ (or $M_\lambda(E,A)$).

**Lemma 2** *For every* $\lambda \in \mathbb{N}$ *there exists a term* $U := s^\mu e_{i_1 j_1}\cdot\dots\cdot e_{i_\mu j_\mu}\cdot a_{k_1\ell_1}\cdot\dots\cdot a_{k_\kappa \ell_\kappa}$ *of* $(sE-A)$ *defining a term* $T := e^\lambda_{i_1 j_1}\cdot\dots\cdot e^\lambda_{i_\mu j_\mu}\cdot a^{\lambda-1}_{k_1\ell_1}\cdot\dots\cdot a^{\lambda-1}_{k_\kappa\ell_\kappa}$ *of* $M_\lambda(E,A)$ *such that*

$$|T| = \mathrm{rk}\, M_\lambda(E,A).$$

**Proof** (Sketch) Choose $U$ as a term that determines the maximum degree of all minors of size $(\mathrm{rk}\, E + \sum_{i=1}^{\lambda-1}\nu_i)$. Using (1) and (2), we obtain the degree of $U$ (i. e. the number of $E$-elements in $U$):

$$\deg U = \mu = \mathrm{rk}\, E - \sum_{i=1}^{\lambda-1}(i-1)\nu_i$$

Now, the statement can be verfied exploiting (3) by direct calculation. ∎

## Main Result

We turn now to structure matrix pencils $[sE - A]$, i. e. we suppose the entries of the matrices $E$ and $A$ are not precisely known. More exactly, we distinguish between two types of entries: entries that are fixed to zero and entries that are assumed to be mutually independent. In this way the real matrices $E$ and $A$ are replaced by binary structure matrices $[E]$ and $[A]$ of the same size.

**Definition 1** *The entries of a structure matrix* $[M]$ *are either fixed to zero or indeterminate values. By fixing all the indeterminate entries of* $[M]$ *to some particular real values we obtain an* admissible realization $M$ *of the binary structure matrix* $[M]$*; for short, we write* $M \in [M]$*. Two matrices* $M' \in [M]$ *and* $M'' \in [M]$ *are called* structurally equivalent.

Each admissible realization $M \in [M]$ where $[M]$ possesses $h > 0$ indeterminate entries can be interpreted as an element of a vectorspace $\mathbb{R}^h$. We say that a matrix property holds structurally for $[M]$ if this property holds for almost all $M \in \mathbb{R}^h$. Here "almost all" means "for all exept for those in some proper algebraic variety in $\mathbb{R}^h$" (comp. [10]). For example, the *structural rank* of $[M]$ is a very important structural property of the set of structurally equivalent matrices. It is defined by

$$\text{s-rk}\,[M] = \max_{M\in[M]} \mathrm{rk}\,(M)$$

Similarly we can define structural properties of matrix pencils $[sE - A]$, taking the two structure matrices $[E]$ and $[A]$ into account. The structure at infinity of structure matrix pencils is a structural property. It can be determined by computing the *generic rank* of $M_\lambda[E,A]$:

$$\text{g-rk}\, M_\lambda[E,A] = \max_{\substack{E\in[E]\\ A\in[A]}} \mathrm{rk} \begin{pmatrix} [E] & & & & \\ [A] & [E] & & & \\ & [A] & [E] & & \\ & & \ddots & \ddots & \\ & & & [A] & [E] \end{pmatrix}$$

for all $\lambda \in \{1, 2, \ldots, n_\nu + 1\}$. Note that we must take the dependencies between the various $[E]$ as well as between the various $[A]$ into account. Therefore the generic rank can differ from the structural rank where *all* entries are considered to be independent:

$$\text{g-rk}\, M_\lambda[E,A] \le \text{s-rk}\, M_\lambda[E,A]$$

Now we are able to formulate our main result.

**Theorem 1** $\forall\lambda\in\mathbb{N}$ : *g-rk* $M_\lambda[E,A] =$ *s-rk* $M_\lambda[E,A]$.

**Proof** (Sketch) First realize that Lemma 2 is also valid for structure matrix pencils. Thus we choose a term $U := s^\mu e_{i_1 j_1}\cdot\dots\cdot e_{i_\mu j_\mu}\cdot a_{k_1\ell_1}\cdot\dots\cdot a_{k_\kappa\ell_\kappa}$ of $[sE - A]$ defining a term $T := e^\lambda_{i_1 j_1}\cdot\dots\cdot e^\lambda_{i_\mu j_\mu}\cdot a^{\lambda-1}_{k_1\ell_1}\cdot\dots\cdot a^{\lambda-1}_{k_\kappa\ell_\kappa}$ of $M_\lambda[E,A]$ with $|T| =$ g-rk $M_\lambda[E,A]$. The term $T$ has a special property: Since all elements $e_{ij}$ of $U$ appear $\lambda$ times in $T$ and all elements $a_{ij}$ of $U$ appear $(\lambda - 1)$ times in $T$, this term $T$ exists exactly once in a certain $|T|$-sized minor of $M_\lambda[E,A]$. This means that the minor under consideration cannot vanish for almost all $E \in [E]$ and $A \in [A]$.

Now we assume s-rk $M_\lambda[E,A] >$ g-rk $M_\lambda[E,A]$. Then there is a term $T'$ of size $|T'| = |T| + 1$ in $M_\lambda[E,A]$. Using $T'$ and $U$, it is always possible to generate a term $U' := s^{\mu'} e_{i'_1 j'_1}\cdot\dots\cdot e_{i'_{\mu'} j'_{\mu'}}\cdot a_{k'_1\ell'_1}\cdot\dots\cdot a_{k'_{\kappa'}\ell'_{\kappa'}}$ of $(sE - A)$ defining a term $T'' := e^\lambda_{i'_1 j'_1}\cdot\dots\cdot e^\lambda_{i'_{\mu'} j'_{\mu'}}\cdot a^{\lambda-1}_{k'_1\ell'_1}\cdot\dots\cdot a^{\lambda-1}_{k'_{\kappa'}\ell'_{\kappa'}}$ with $|T''| > |T|$. The generation of the term $U'$ is quite a long procedure, applying graph-theoretic tools. Due to lack of space, we omit this part of the proof. The whole procedure can be found in [9].

Now we use the same arguments for $T''$ as for $T$ to prove that a $|T''|$-sized minor of $M_\lambda[E,A]$ containing $T''$ cannot vanish for almost all $E \in [E]$ and $A \in [A]$. Thus $|T''| \le$ g-rk $M_\lambda[E,A] = |T|$, contradicting $|T''| > |T|$. ∎

## Computational Results

In the previous sections we have shown that the structure at infinity of structure matrix pencils $[sE-A]$ can be obtained by determining the structural rank of $M_\lambda[E,A]$ for all $\lambda \in \{1,2,\ldots,n_\nu+1\}$. Bipartite-matching algorithms are very efficient for computing the structural rank of matrices (cf. [1]). The complexity bound for the computational effort of bipartite-matching problems is known to be $O(\sqrt{n}m\log(n^2/m)/\log n$ , where $n$ is the sum of rows and columns, and $m$ is the number of nonzero entries of the matrix.

Since the index (i. e. the size $n_\nu$ of the largest Jordan block in $N$) of $[sE-A]$ is smaller than a number $k$ in all practical cases (say $k=4$), the bound mentioned above is also valid for the determination of the structure at infinity.

Table 1 shows the computational results for index-1 and index-2 problems, using a Sun UltraStation 1 Model 170 workstation. We applied a C implementation of a push-relabeled algorithm (available on http:// www.cs.sunysb.edu/ ˜algorith/ implement/ bipm/ implement.shtml) as a basis for our infinite-structure algorithm. 1% of the entries of the $n \times n$ structure matrices $[E]$ and $[A]$ are chosen as nonzero elements. The last column of Table 1 shows the practical complexity $c$ of the computational effort depending on the dimension $n$. The results show that our method is well suited even for large systems.

| Dimension | 500 | 1000 | 2000 | 4000 | c |
|---|---|---|---|---|---|
| Index 1 | 0.18 | 0.47 | 1.80 | 6.64 | 1.8 |
| Index 2 | 0.28 | 1.14 | 4.20 | 14.67 | 1.9 |

Table 1: Computational Results (in secs.)

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