Stabilization of Linear Dynamical Systems with Scalar Quantizers under Communication Constraints

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***Abstract*—This paper addresses a feedback stabilization problem for linear time-invariant dynamical systems where the feedback control loop is closed over a noiseless time-variant and rate-limited communication link. In contrast to the previous work, we assume a set of scalar quantizers and propose a method for stabilizing the system at reduced data rates.**

I. Introduction

Historically, communication and control have been separate research areas with more or less independent theories. Recently, however, there has been an increasing demand on networks consisting of control and communication systems which are subject to uncertainty and limited time-varying channel capacity. In such applications, due to finite capacity, the system state cannot be represented with high precision at the output of the communication channel, and only a distorted version of system state or system output is available for feedback. Therefore, the fundamental questions raised here is to find encoders, decoders and controllers to achieve certain performance objectives associated with the control and communication subsystems. In this paper, the primary performance objective is stabilization of the dynamical system with simple scalar quantization schema and, once the system is stabilizable, the main problem is how to ensure stabilization with data rates as small as possible. The problem is of great interest in wireless sensor networks where the energy consumption for transmission is to be minimized in order to maximize the network life time.

Various publications have introduced necessary and sufficient conditions for observability and stabilizability of such a basic communication/control systems in various senses [1], [2], [3], [4], [5], [6]. These conditions are often given in the form of a lower bound on the channel capacity in terms of rate of the change of dynamical system. In particular, it is shown [2] that the minimum capacity required for achieving observability and stabilizability of linear time-invariant discrete-time plants is given by

$$R_g := \sum_i \max\{0, \log_2 |\lambda_i|\}, \tag{1}$$

where λ_i are an eigenvalue of the system's coefficient matrix.

A. Paper contribution

This paper generalizes previous work in three directions:
1. In contrast to [2], we assume a set of scalar quantizers (one for each system state variable) and consider the problem of optimal quantization encoding for noiseless control system with perfect system state observation at the system output in the sense of minimizing the difference of Lyapunov functions. Surprisingly, it turns out that an optimal encoding (optimal bit allocation) is independent of actual system state, which makes the scheme amenable to practical implementations. Further, we provide a lower bound on the bit rate which is sufficient to achieve the stabilization under scalar quantizers. This bound quantifies the amount of extra bits that are required if scalar quantizers are used instead of an optimal vector quantizers. Finally, we will take into account variations in the capacity of the communication link.
2. We extend the results to noisy control systems with imperfect system state observation at the system output by estimating the uncertainty set under Kalman filtering.
3. We provide a heuristic method to stabilize system with minimum bit rate.

The proofs are omitted due to the lack of space.

II. System Model

We consider a feedback control system consisting of a plant, system state estimator and controller, in which the information about the system states is communicated to the controller via finite-capacity wireless communication link (including one transmitter and one receiver). The underlying system model is illustrated in Figure 1. The dynamics of a plant to be controlled evolves according to a *discrete-time linear system* of the form:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{w}(k) \tag{2a}$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{v}(k) \tag{2b}$$

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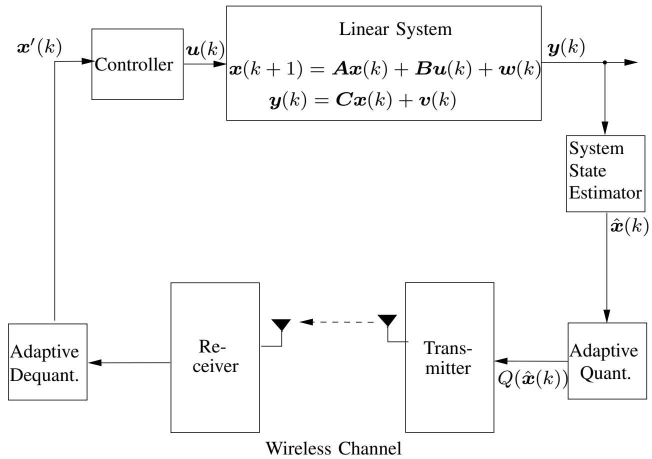

Fig. 1. The underlying system model

with $k \in \mathbb{N}_+$, $\boldsymbol{x}(k), \boldsymbol{w}(k) \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{u}(k) \in \mathbb{R}^m$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{y}(k), \boldsymbol{v}(k) \in \mathbb{R}^q$, $\boldsymbol{C} \in \mathbb{R}^{q\times n}$ and $n, m, q \in \mathbb{N}$.[1] Here and hereafter, $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{u}$ are the system state, system output and system input, respectively; the vector $\boldsymbol{w}$ denotes the disturbance to the system and it is modelled as a zero-mean random variable distributed according to some given probability distribution function; $\boldsymbol{v}$ is a random variable representing the noise attached to the observation, we assume the distribution of $\boldsymbol{v}$ has bounded support.

The system state $\boldsymbol{x}$ is in general not directly observed at the system output. Therefore the full system state information must be estimated using system output $\boldsymbol{y}$ over time by system state estimator. Since the system estimated system state $\hat{\boldsymbol{x}}$ is a real-valued variable, it must be quantized prior to transmission via the communication link due to constraints on transmission rates. The quantization is performed by an adaptive quantizer whose output is fed to the transmitter. At the receiver side, an adaptive dequantizer attempts to reconstruct the estimate and provides the receiver-side system state estimate $\boldsymbol{x}'$ to the controller. We assume that the system input $\boldsymbol{u}$ is the image of $\boldsymbol{x}'$ under a linear map:

$$\boldsymbol{u}(k) = \boldsymbol{K}\boldsymbol{x}'(k), \quad k \in \mathbb{N} \tag{3}$$

where $\boldsymbol{x}'(k)$ is the received system state estimate at time point k and $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ represents a time-invariant linear controller.

Throughout the paper, the wireless communication link is modelled as an error-free finite-capacity communication channel whose capacity may vary over time. The time variations should capture inherent fading effects in wireless communications channels, while the assumption of an error-free channel requires the use of appropriate coding strategies including forward error correction (FEC), ARQ, etc.

[1]Throughout the paper, $\mathbb{N}, \mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of nonegative integers, positive integers, real numbers and nonnegative real numbers, respectively.

III. Stabilization with Scalar Quantizers for A Noise-free System with Perfect System State Observation

Let us first consider stabilization problem of a noise-free control system with perfect system state observation at the output of the system, that is to say, we have $\boldsymbol{y}(k) = \boldsymbol{x}(k)$ and $\boldsymbol{w}(k) = 0$ in this case.

A. Scalar Quantization

Different quantization designs were proposed (see for instance [7], [8], [2]). In particular, Reference [2] proposed a quantization scheme that ensures stability of a control system in which the communication link operates at a data rate given by the eigenvalue rate condition (1). This scheme is however based on vector quantization, and therefore has a prohibitively high computational complexity if the number of system states n is large, in which case it is not amenable to practical implementations. In order to reduce the complexity, this paper considers the possibility of using a set of scalar quantizers, each for one system state variable.

Let $q_i : \mathbb{R} \to \mathbb{R}$ be a uniform mid-tread quantizer for the ith system state variable $x_i, 1 \le i \le n$. We have

$$q_i(x_i) = \begin{cases} \lfloor \frac{l_i - c_i}{2\Delta_i} + \frac{1}{2} \rfloor \Delta_i + c_i, & \text{if } x_i - c_i \ge l_i/2 \\ -\lfloor \frac{l_i - c_i}{2\Delta_i} + \frac{1}{2} \rfloor \Delta_i + c_i, & \text{if } x_i - c_i < -l_i/2 \\ \lfloor \frac{x_i - c_i}{\Delta_i} + \frac{1}{2} \rfloor \Delta_i + c_i & \text{otherwise} \end{cases}$$

with $l_i \in \mathbb{R}_+$, $c_i \in \mathbb{R}$, $\Delta_i = l_i/M_i$ (M_i is an odd integer due to the symmetry). Here and herafter, c_i is the centroid and $[-l_i/2 + c_i, l_i/2 + c_i]$ is the interval, in which the quantizer is not saturated. M_i is the number of the quantization levels and $\Delta_i = l_i/M_i$ is the minimum quantization interval. As aforementioned, we have n such scalar quantizers for each system state variable so that, given some $\boldsymbol{x}(k)$, the quantization process (denoted by Q) is a map from $\mathbb{R}^n$ into $\mathbb{R}^n$ defined to be $Q(\boldsymbol{x}(k)) = (q(x_1(k)), \ldots, q(x_n(k)))^T, k \in \mathbb{N}$.

Let $\boldsymbol{l} = (l_1, \ldots, l_n), \boldsymbol{c} = (c_1, \ldots, c_n)$ and $\boldsymbol{M} = (M_1, \ldots, M_n)$ and note that these parameters may vary with time so that the quantizer Q depends on k. In what follows, we use $\boldsymbol{l}(k), \boldsymbol{c}(k)$ and $\boldsymbol{M}(k)$ to denote the values of these parameters at time k. Let $R_i = \log_2 M_i$ with $R_i(k)$ being the value of R_i at time k. Then, for any $k \in \mathbb{N}$, $R_i(k)$ bits are used to encode one system state $x_i(k)$. The vector $\boldsymbol{R} = (R_1, \ldots, R_n) \in \mathbb{R}^n_+$ is called *rate or bit allocation* and $\boldsymbol{R}(k)$ is the value of $\boldsymbol{R}$ at time k.

Assuming that none of the n scalar quantizers is saturated, in which case $|x_i(k) - c_i(k)| \le l_i(k)/2$ for all $i = 1, \ldots, n$, the quantization error $\boldsymbol{e}(k) := Q(\boldsymbol{x}(k)) - \boldsymbol{x}(k)$ is bounded by

$$\|\boldsymbol{e}(k)\| \le \Big(\sum\nolimits_i \Delta_i^2(k)\Big)^{\frac{1}{2}}/2 \tag{4}$$

with $\Delta_i(k) = l_i(k)/2^{R_i(k)}$, $k \in \mathbb{N}$. Given k, let $\mathcal{L}(\boldsymbol{l}(k), \boldsymbol{c}(k))$ be the compact subset of the system state space where the quantizer Q is not saturated.

B. Uncertainty Set of System

Due to the rate limitations, there is an inherent uncertainty about the system state at the controller. The set of all possible values of the actual system state is called *uncertainty set* of the system state. If this uncertainty could be removed at the controller, we would have a classical control problem. Now the key idea is to remove this uncertainty asymptotically as $k \to \infty$ using suitable dynamic quantization.

To be more precise, let Ω_0 be the initial uncertainty set which is assumed to be compact with non-empty interior and known to the quantizer and dequantizer. The quantizer sends an index of the quantizer cell containing the actual system state to the dequantizer, which can decrease the uncertainty about the system state to one quantization cell denoted by Ω_0'. Now, given Ω_0', the uncertainty set of the system state for the next time Ω_1 is predicted at the quantizer and dequantizer. Note that Ω_1 certainly contains the system state $\boldsymbol{x}(1)$ as it contains all possible system states evolving from the system states in Ω_0', and therefore in particular from $\boldsymbol{x}(0) \in \Omega_0'$. The whole procedure is repeated with Ω_1, which is illustrated in Figure 2. We will show that both quantizer and dequantizer

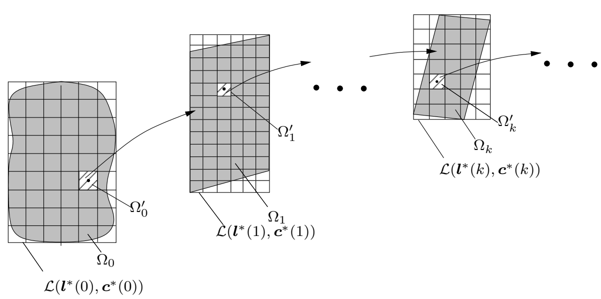

Fig. 2. Evolution of Uncertainty Sets.

can predict the uncertainty sets independently so that Ω_k and Ω_k' are known to both sides of the communication link for all k, provided that there is the transmission is error-free, which is true by the assumption. If the transmission rate is sufficiently large, we will show that the uncertainty about the system state disappears as $k \to \infty$.

The following two questions arise immediately: 1) How to choose the quantizer-dequantizer pair (including $\mathcal{L}(\boldsymbol{l}, \boldsymbol{c})$ and the rate allocation $\boldsymbol{R}$) and 2) how to predict the uncertainty set Ω_{k+1} when Ω_k' is given. In what follows, we are going to address these questions.

C. Scalar Quantization Design and Bit Allocation

Under the assumption of an error-free communication link, the control signal $\boldsymbol{u}$ is of the form (3) where $\boldsymbol{x}' = Q(\boldsymbol{x})$ is the quantized system state. It is assumed that the controller $\boldsymbol{K}$ is chosen such that $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is stable (i.e. $\forall_i |\lambda_i(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})| < 1$), meaning that $\boldsymbol{K}$ stabilizes (2) in the classical deterministic setting. Since

$$Q(\boldsymbol{x}(k)) = \boldsymbol{x}(k) + \boldsymbol{e}(k),$$

where $\boldsymbol{e}(k) \in \mathbb{R}^n$ is the quantization error at time k, it follows from (2) that the system is governed by the following equation:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) \\ &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{K}Q(\boldsymbol{x}(k)) \\ &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(k)\,. \end{aligned} \tag{5}$$

For our analysis of system stability, let $V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ with a positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ be a candidate for a Lyapunov function. The difference of the values of the Lyapunov function at two successive time points $\Delta V(\boldsymbol{x}(k))$ is an important "measure" of the system stability. Intuitively, minimizing $\Delta V(\boldsymbol{x}(k))$ means that the system is stabilized as fast as possible for some $\boldsymbol{P}$. Considering (5) yields

$$\begin{aligned} \Delta V(\boldsymbol{x}(k)) &= \boldsymbol{x}^T(k+1)\boldsymbol{P}\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \\ &= \boldsymbol{x}^T(k)((\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}) - \boldsymbol{P})\boldsymbol{x}(k) \\ &\quad + 2\boldsymbol{x}^T(k)(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(k) \\ &\quad + \boldsymbol{e}^T(k)\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\boldsymbol{e}(k) \end{aligned}$$

In all that follows, let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be symmetric positive definite matrices such that $(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}) - \boldsymbol{P} + \boldsymbol{Q} = 0$. Note that as $\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ is stable (by assumption), such matrices exist [9]. With this choice of $\boldsymbol{P}$ and $\boldsymbol{Q}$ and with (4), we can bound $\Delta V(\boldsymbol{x}(k))$ as follows

$$\begin{aligned} &\Delta V(\boldsymbol{x}(k)) \le \Delta V'(\boldsymbol{x}(k)) \\ &:= \|(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\| \Big(\sum\nolimits_i \Delta_i(k)^2\Big)^{\frac{1}{2}} \|\boldsymbol{x}(k)\| \\ &\quad + \frac{\|\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\|}{4}(\sum\nolimits_i \Delta_i(k)^2) - \lambda_{min}(\boldsymbol{Q})\|\boldsymbol{x}(k)\|^2 \end{aligned} \tag{6}$$

where $\lambda_{min}(\boldsymbol{Q}) > 0$ denotes the smallest eigenvalue of $\boldsymbol{Q}$. This bound is valid if none of the quantizers is saturated, i.e., $\boldsymbol{x}(k) \in \mathcal{L}(\boldsymbol{l}(k), \boldsymbol{c}(k))$. Note that $\Delta V'(\boldsymbol{x}(k))$ depends on $\boldsymbol{R}$, $\boldsymbol{l}$ and $\boldsymbol{x}(k)$. We often write $\Delta V'(\boldsymbol{x})$ for brevity.

We now design a quantizer Q, which depends on k, that minimizes $\Delta V'(\boldsymbol{x})$ at every time step under the rate constraints: $\forall_k \sum_i R_i(k) \le R_{tot}(k)$, where $R_{tot}(k)$ is the maximum number of bits that may be communicated over the channel at time k, which is assumed to be known at both sides of the channel. In practice, $2^{R_i(k)}$ or even $R_i(k)$ need to be integers, a constraint we will neglect throughout. Put in a more formal way, the problem is:

Problem 1. *Given $\boldsymbol{x} \in \mathbb{R}^n$, $R_{tot} \in \mathbb{R}_+$, and $\Omega \subseteq \mathbb{R}^n$ compact with non-empty interior,*

$$\begin{aligned} &\textit{minimize}\ \Delta V'(\boldsymbol{x}) \\ &\qquad \textit{s.t.}\ R_1 + \cdots + R_n \le R_{tot} \quad \textit{and} \quad \Omega \subseteq \mathcal{L}(\boldsymbol{l}, \boldsymbol{c}) \end{aligned}$$

in variables $\boldsymbol{l}, \boldsymbol{R} \in \mathbb{R}^n_+$ and $\boldsymbol{c} \in \mathbb{R}^n$.

Note that this problem must be solved at both quantizer and dequantizer independently, although the dequantizer has no direct access to $\Delta V'(\boldsymbol{x}(k))$. The constraint $\Omega_k \subseteq \mathcal{L}(\boldsymbol{l}(k), \boldsymbol{c}(k))$ ensures that the quantizer is not saturated, which implies $\boldsymbol{x}(k) \in \mathcal{L}(\boldsymbol{l}(k), \boldsymbol{c}(k))$. The latter condition cannot be used since only Ω_k is known to both the quantizer and dequantizer.

The above problem can be divided into smaller problems that can be solved separately. We first observe that components $\boldsymbol{c}$ and $\boldsymbol{l}$ of solutions $(\boldsymbol{l}, \boldsymbol{c}, \boldsymbol{R})$ to Problem 1 are independent of both $\boldsymbol{x}$ and R_{tot}, which is quite intuitive.

Proposition 1. *Let $\boldsymbol{x}$, R_{tot} and Ω be as in Problem 1, assume $\boldsymbol{BK} \neq 0$, and let $(\boldsymbol{l}, \boldsymbol{c}, \boldsymbol{R})$ solve Problem 1. Then $(\boldsymbol{l}, \boldsymbol{c})$ is uniquely determined by*

$$(\boldsymbol{l}, \boldsymbol{c}) = \underset{(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{c}})}{\arg\min}\ \text{volume}\{\mathcal{L}(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{c}}) \subset \mathbb{R}^n | \Omega \subseteq \mathcal{L}(\tilde{\boldsymbol{l}}, \tilde{\boldsymbol{c}})\}.$$

In particular, if Ω is the image of $\mathcal{L}(\boldsymbol{l}', \boldsymbol{c}')$ under a non-singular matrix A, $\Omega = A\mathcal{L}(\boldsymbol{l}', \boldsymbol{c}')$, then

$$\boldsymbol{l} = |A|\boldsymbol{l}', \quad \boldsymbol{c} = (A + BK)\boldsymbol{c}',$$

where $|A|$ denotes the matrix with entries $|A_{i,j}|$.

We now determine bit rates $R_i(k)$ that minimize $\Delta V'(\boldsymbol{x}(k))$ when $\boldsymbol{l}(k)$ and $\boldsymbol{c}(k)$ are already known.

Proposition 2. *Let $\boldsymbol{x}$, R_{tot} and Ω be as in Problem 1, assume $\boldsymbol{BK} \neq 0$, the optimal R_i^* has the form*

$$R_i^* = \max\{0, w - \log_2 \frac{1}{l_i}\}$$

with some $w \in \mathbb{R}$ satisfying $\sum_i \max\{0, w - \log_2 \frac{1}{l_i}\} = R_{tot}$.

The special structure of the solution suggests an efficient method of solution. In fact, it can be solved by a water-filling procedure [10]. The solution, roughly speaking, would allocate most bits to the most "uncertain" system states, i.e., to those states $x_i(k)$ for which $l_i(k)$ is large.

For later reference, we summarize the results of this section.

Corollary 1. *Let $\boldsymbol{x}$, R_{tot} and Ω be as in Problem 1, assume $\boldsymbol{BK} \neq 0$ and $R_{tot} \geq 0$. Then Problem 1 has a unique solution $(\boldsymbol{l}, \boldsymbol{c}, \boldsymbol{R})$, which is given by Propositions 1 and 2. That solution does not depend on $\boldsymbol{x}$, and its components $\boldsymbol{l}$ and $\boldsymbol{c}$ do not depend on R_{tot} either.*

D. Stability of Closed Loop

Now let us look at the stability of the closed loop under the proposed quantization. The solution of (5) is given by

$$\boldsymbol{x}(k) = (\boldsymbol{A} + \boldsymbol{BK})^k \boldsymbol{x}(0) + \sum_{i=0}^{k-1} (\boldsymbol{A} + \boldsymbol{BK})^{k-1-i} \boldsymbol{BK} \boldsymbol{e}(i)\,.$$

Hence, if $\boldsymbol{A} + \boldsymbol{BK}$ is stable, the state $\boldsymbol{x}(k)$ vanishes as $k \to \infty$ if the error $\boldsymbol{e}(k)$ does [2, Lemma 5.1]. The latter can not, in general, be guaranteed if a traditional (non-adaptive) quantizer is used [1, Proposition 2.1]. The main question here is under which conditions on $R_{tot}(k)$, scalar quantization using adaptive bit allocation as proposed in this paper would guarantee $\lim_{k\to\infty} \boldsymbol{e}(k) = 0$. We present a sufficient condition under which, roughly speaking, the uncertainty sets keep shrinking uniformly for all time.

Theorem 1. *Let $\boldsymbol{x}(0) \in \Omega_0$, $\Omega_0 \subseteq \mathbb{R}^n$ compact with non-empty interior, and let R_{tot} be a sequence of reals. Of the closed loop (5), assume that $\boldsymbol{BK} \neq 0$, $\boldsymbol{A} + \boldsymbol{BK}$ is stable, and $\boldsymbol{A}$ is non-singular with*

$$R_{tot}(k) \geq \alpha > \sum_{i=1}^{n} \max\{0, \log_2 \gamma_i\} \tag{7}$$

for all k and some α, where $\gamma_i = \sum_{j=1}^n |A_{i,j}|$. Finally, let the quantizer Q of (5) be determined by solutions $(\boldsymbol{l}(k), \boldsymbol{c}(k), \boldsymbol{R}(k))$ of Problem 1 with $\boldsymbol{x}(k)$ and $R_{tot}(k)$ substituted for $\boldsymbol{x}$ and R_{tot}, respectively. Then $\lim_{k\to\infty} \boldsymbol{x}(k) = 0$ for the closed loop (5).

Let us compare the bound (7) with the capacity bound (1), which is sufficient for system stabilization under optimal vector quantization [2]. We assume here that all eigenvalues of $\boldsymbol{A}$ have magnitudes greater than or equal to 1.

Let $\boldsymbol{S}_A$ be the diagonal matrix defined by $(S_A)_{i,i} = 1/\max\{1, \gamma_i\}$ and set $R_b = -\log_2 \det \boldsymbol{S}_A$. According to Theorem 1, any rate larger than R_b is sufficient for the proposed quantization method to stabilize the control system. If $\boldsymbol{Y}_A = \boldsymbol{S}_A \boldsymbol{A}$, then $|\det \boldsymbol{Y}_A| = |\det \boldsymbol{S}_A|\,|\det \boldsymbol{A}| = \frac{|\det \boldsymbol{A}|}{2^{R_b}} = 2^{R_g - R_b}$, and $|\det \boldsymbol{Y}_A| \leq 1$ as $\rho(\boldsymbol{Y}_A) \leq \|\boldsymbol{Y}_A\|_\infty \leq 1$. Hence, $R_b \geq \log_2 \det |\boldsymbol{A}| = R_g$ and

$$R_b = \log_2 \left| \frac{\det \boldsymbol{A}}{\det \boldsymbol{Y}_A} \right| = R_g - \log_2 |\det \boldsymbol{Y}_A|\,.$$

Thus, $-\log_2 |\det \boldsymbol{Y}_A|$ is the price to pay in terms of extra bit rate needed for stabilization when computationally simpler scalar quantizers are used instead of a vector quantizers.

IV. Uncertainty Set Estimation for Noisy System

So far we have assumed that the system is noiseless and the system state is fully observable at the system output. This section extends the results to systems with disturbances and imperfect system state observation. Our system is discribed by equation (2). The classical Kalman Filter is used as a system state estimator [11], [12]. Now the partially observed linear quadratic stochastic control problem (called LQG) is separated into two optimization problems: optimal estimation problem (with respect to MMSE criterion) and deterministic linear quadratic optimal control problem (LQR).

The main question is how to estimate the uncertainty set in this case. As in the noiseless case, the key idea is to make quantization error vanish along with time by predicting the increment of the uncertainty set of the system state based on the system dynamics and by decreasing the uncertainty set with the received information. The essential difference compared to the noiseless case is that, instead of the actual system state $\boldsymbol{x}(k)$, an *estimated system state* $\hat{\boldsymbol{x}}(k)$ is transmitted that evolves according its own dynamics. So the method for decreasing uncertainty in this case is the same as described in section III-B except that the uncertainty set of estimated system state $\hat{\boldsymbol{x}}$ must vanish with time as well.

As the prediction problem is more complicated, we have to deal with a more general structure of the uncertainty set. In this paper, the uncertainty set $\hat{\Omega}'_k$ for the estimated system state

$\hat{\boldsymbol{x}}$ is assumed to be a convex polyhedron, i.e., $\hat{\Omega}'_k = \mathcal{P}(\boldsymbol{M}, \boldsymbol{d})$,

$$\mathcal{P}(\boldsymbol{M}, \boldsymbol{d}) := \{\boldsymbol{x} | \boldsymbol{M}\boldsymbol{x} \leq \boldsymbol{d}\}$$

for some $\boldsymbol{M} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{d} \in \mathbb{R}^m$. The point is that images of $\mathcal{P}(\boldsymbol{M}, \boldsymbol{d})$ under non-singular linear maps can be easily determined from $\boldsymbol{M}$ and $\boldsymbol{d}$ [13, Lemma III.6].

In what follows, we use the following notation $\boldsymbol{V} = E\{\boldsymbol{v}(k)\boldsymbol{v}(k)^T\}$, $\hat{\boldsymbol{x}}_{k|s} := E\{\boldsymbol{x}(k) \mid \boldsymbol{y}(0), \ldots, \boldsymbol{y}(s))\}$ and $\Sigma_{k|s} := E\{(\boldsymbol{x}(k) - \hat{\boldsymbol{x}}_{k|s}))(\boldsymbol{x}(k) - \hat{\boldsymbol{x}}_{k|s})^T\}$. Start with prior mean and error covariance $\hat{\boldsymbol{x}}_{0|-1} = \hat{\boldsymbol{x}}_{-1|-1} = E\{\boldsymbol{x}(0)\}$ and $\Sigma_{0|-1} = \boldsymbol{X}_0$, for $t = 0, \ldots, N$. Define $\boldsymbol{L}_k = \Sigma_{k|k-1}\boldsymbol{C}^T(\boldsymbol{C}\Sigma_{k|k-1}\boldsymbol{C}^T + \boldsymbol{V})^{-1}$ and set $\hat{\boldsymbol{x}}(0) = \boldsymbol{L}_0\boldsymbol{y}(0)$. Then, it can be shown [11], [12] that the estimated system state $\hat{\boldsymbol{x}}$ under the Kalman filter satisfies

$$\begin{aligned} \hat{\boldsymbol{x}}(k+1) =& (\boldsymbol{I} - \boldsymbol{L}_{k+1}\boldsymbol{C})(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\hat{\boldsymbol{x}}(k) \\ &+ \boldsymbol{L}_{k+1}\boldsymbol{x}(k+1) + \boldsymbol{L}_{k+1}\boldsymbol{v}(k+1) \end{aligned} \tag{8}$$

where $\hat{\boldsymbol{x}}(k) := \hat{\boldsymbol{x}}_{k|k}$. The following proposition provides the solution of this estimation problem.

Proposition 3. *Let $\hat{\boldsymbol{x}}$ satisfy (8) at time k and let $\boldsymbol{v}(k)$ be a bounded random variable with $\|\boldsymbol{v}(k)\|_\infty \leq V$ for all k. If $\hat{\boldsymbol{x}} \in \hat{\Omega}'_k = \mathcal{P}(\boldsymbol{M}, \boldsymbol{d})$, then, for arbitrarily small $\varepsilon > 0$,*

$$\Pr\left[\hat{\Omega}_{k+1} \subseteq \mathcal{P}(\boldsymbol{M}\boldsymbol{G}^{-1}, \boldsymbol{d} + V\boldsymbol{s} + \boldsymbol{h})\right] > 1 - \varepsilon$$

where $\boldsymbol{G} = (\boldsymbol{I} - \boldsymbol{L}_{k+1}\boldsymbol{C})(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})$ is invertable, the i-th entries of $\boldsymbol{s} \in \mathbb{R}^n$ and $\boldsymbol{h} \in \mathbb{R}^n$ are $s_i = \|(\boldsymbol{M}\boldsymbol{G}^{-1}\boldsymbol{L}_{k+1})_{i*}\|_2$ and $h_i = \|(\boldsymbol{M}\boldsymbol{G}^{-1}\boldsymbol{L}_{k+1}\boldsymbol{C})_{i*}\|_2(\|\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\|_2 \max_{\hat{\Omega}_k} \|\hat{\boldsymbol{x}}(k)\|_2 + \|\boldsymbol{\epsilon}(k)\|_2)$, respectively, and $\boldsymbol{\epsilon}(k) \in \mathbb{R}^n$ with $\epsilon_i = \sqrt{\Sigma_{k+1|k,i}/\varepsilon}$.

By the proposition, given $\hat{\Omega}_k$, we can predict the uncertainty set $\hat{\Omega}_{k+1}$ on both sides of the communication link by using the system equation (8). Therefore, the quantization scheme is similar to that for the noiseless case except that the underlying dynamical system is that of the estimated system state $\hat{\boldsymbol{x}}(k)$.

V. Stabilization with Minimum Bit Rate

Our goal so far has been to minimize $\Delta V(\boldsymbol{x})$ subject to some constraints. Alternative strategy is to reduce the transmission rate by keeping the values of $\Delta V(\boldsymbol{x})$ negative, in which case the system is driven to its equilibrium point with a low effort. Such an approach is of interest to numerous applications in wireless sensor networks, where the primary objective is to save the energy consumption at sensor nodes.

The following proposition provides a basis for the design of heuristic approaches to the problem.

Proposition 4. *Consider (5) and suppose that $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ is stable. Let $\boldsymbol{T} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}$ and*

$$\Phi = \frac{1}{2\lambda_{min}(\boldsymbol{Q})}(\|\boldsymbol{T}\| + \sqrt{\|\boldsymbol{T}\|^2 + \lambda_{min}(\boldsymbol{Q})\|\boldsymbol{K}^T\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}\|}).$$

If there is a quantizer-dequantizer pair such that the following constraints hold for all k with some (sufficiently small) $\epsilon > 0$:

$$\Phi \cdot \sqrt{\sum\nolimits_i \left(l_i(k)/2^{R_i(k)}\right)^2} \leq \|\boldsymbol{x}(k)\| - \epsilon \tag{9a}$$

$$\mathcal{L}(\boldsymbol{l}(k), \boldsymbol{c}(k)) \ni \boldsymbol{x}(k) \tag{9b}$$

then $\Delta V(\boldsymbol{x}(k)) < 0$ for all k and $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k)$ is strictly decreasing with respect to k.

This proposition is an immediate consequence of (6). Roughly speaking, if the quantizer parameters $\boldsymbol{l}$ and $\boldsymbol{R}$ are chosen to fulfill the constraints (9a) and (9b) at each step, then $\Delta V(\boldsymbol{x})$ is negative for all k and the system state approaches the equilibrium point arbitrarily closely. In particular, $\boldsymbol{l}$ in (9a) can be chosen to be equal to $\boldsymbol{l}^*$ specified by Proposition 1.

Now since $\|\boldsymbol{R}(k)\|_1 \leq R_{tot}(k)$ must hold for all $k \in \mathbb{N}$, the problem can be formulated as follows:

$$\begin{aligned} &\min \|\boldsymbol{R}(k)\|_1 \\ &\text{s.t. } \Phi \cdot \left(\sum\nolimits_i (l_i(k)/2^{R_i(k)})^2\right)^{1/2} \leq \|\boldsymbol{x}(k)\| - \epsilon && (10a) \\ &\qquad \|\boldsymbol{R}(k)\|_1 \leq R_{tot}(k), R_i(k) \geq 0 \; i = 1, \ldots, n && (10b) \\ &\qquad \Omega_k \subseteq \mathcal{L}(\boldsymbol{l}(k), \boldsymbol{c}(k)) \end{aligned}$$

for some sufficiently small $\epsilon > 0$. It must be emphasized that due to (10b), this problem is *not* always feasible. Indeed, if $R_{tot}(k)$ is small enough, there are no $\epsilon > 0$ and $\boldsymbol{R}(k)$ for which (10a) is satisfied. In such cases, however, we can switch to the previous transmission strategy to minimize $\Delta V(\boldsymbol{x})$.

An optimal rate allocation can found using Lagrangian theory. It may be shown that an optimal rate allocation (in the sense of the above problem) is of the form

$$R_i^* = \max\{0, 1/2 \log \lambda^* \gamma l_i^2\}$$

where $\gamma = \Phi \ln 2(\sum_i (l_i/2^{R_i^*})^2)^{-\frac{1}{2}}$ and $\lambda^* > 0$. The exact solution can be calculated using a water-filling algorithm.

It is important to mention that in contrast to Problem 1, the optimal rate allocation here *does* depend on the actual system states $\|\boldsymbol{x}\|$ so that only the quantizer can calculate the optimal rate allocation. In practice, however, there is a finite number of possible transmission rates (achievable with different coding or modulation strategies). So, the quantizer can inform the dequantizer about the transmission rate together with the quantized information so that the dequantizer can reconstruct the system states correctly.

VI. Simulation Results

We consider a noiseless system with perfect system state observation given by

$$\boldsymbol{A} = \begin{pmatrix} 0.94 & 20.12 & 0.06 & 0.08 \\ -0.05 & 10.1 & -0.05 & 0 \\ 0.26 & -5.12 & 1.14 & -0.08 \\ -0.04 & -10.12 & -0.06 & 0.82 \end{pmatrix}$$

and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 & 4.5 \end{pmatrix}^T$. The coefficient matrix $\boldsymbol{A}$ is unstable with eigenvalues approximately 10, 1.3, 0.9 and 0.8, and the feedback $\boldsymbol{K} = (-0.0354, -11.198, -0.0398, 0.0016)$

stabilizes the system in the classical, deterministic setting. In particular, $\boldsymbol{A} + \boldsymbol{BK}$ is stable with spectral radius approximately 0.94. Here, the rate (1) necessary for stabilization is 4 bits at every time instance, while, according to Theorem 1, our method of scalar quantization using adaptive bit allocation would require 14 bits. In what follows we consider the total bit rate $R_{tot}(k)$ an independent random variable which is uniformly distributed on $[10, 20]$ for every k. The initial system states are equal and chosen randomly from $[-2, 2]$.

A. System Stabilization with Different Rate Allocations

Figure 3 shows the system evolution under different quantization methods. In the top plot, the rates are allocated uniformly among the system states, while an optimal rate allocation (in the sense of Problem 1) was used in the simulation depicted in the bottom plot. Under an optimal rate allocation, the system with converges significantly faster.

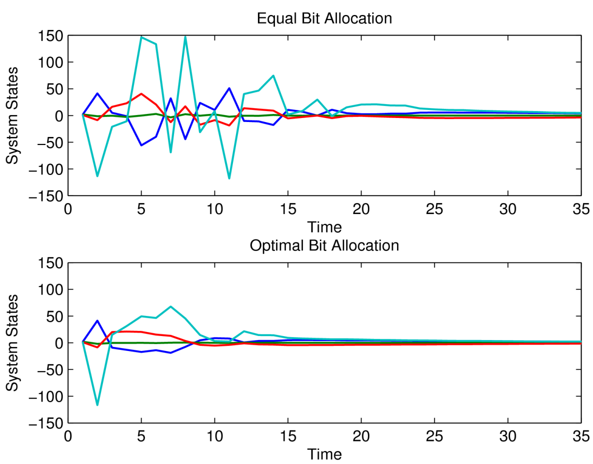

Fig. 3. System Evolution with Different Bit Allocation

B. Stabilization with Minimum Bit Rate

The two plots in Figure 4 shows the system evolution with different transmission strategies. The top plot shows the system behavior when $\Delta V(\boldsymbol{x})$ is minimized (as in Problem 1)), whereas the bottom plot uses a heuristic strategy discussed in the previous section (which is optimal in the sense of Problem (10)).

Finally, Figure 5 depicts an histogram of the transmission rates that are allocated under the two different quantization strategies. We can observe a trade-off between the convergence speed of the system and the number of bits that are used for transmission.

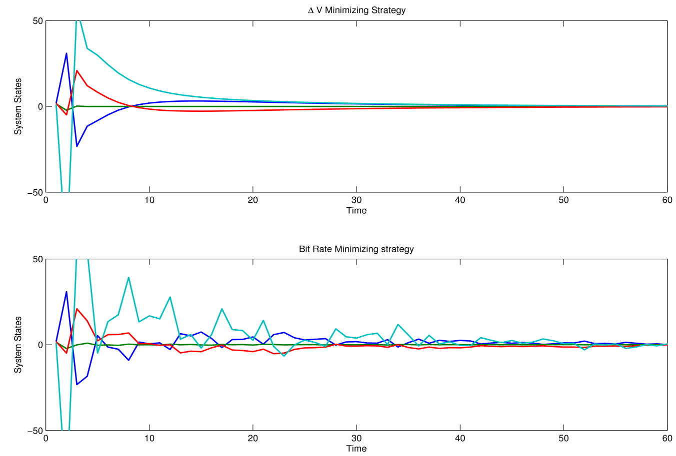

Fig. 4. System Evolution with Different Transmission Strategies

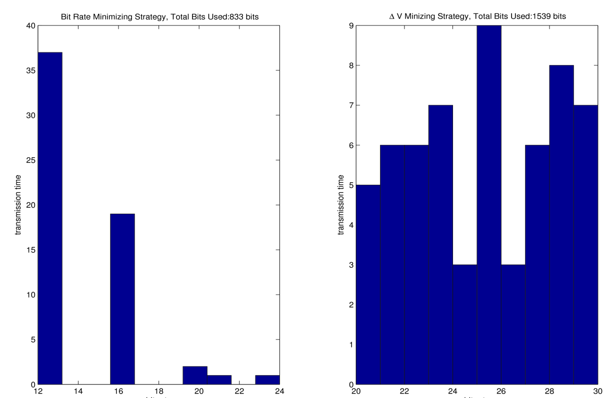

Fig. 5. Bit Rate Used for Different Strategies

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