Impasse Points: Examples and Counterexamples

Gunther Reißig

19 July 1993

Abstract

In this paper we define classes of impasse points, an important phenomenon of Differential-Algebraic Equations (DAEs). Focussing on DAEs occuring in the analysis of electrical networks, a brief discussion of properties of these classes and examples are given. Some of these examples are counterexamples to published assertions on impasse points.

Als Manuskript gedruckt.

Technische Universität Dresden

Herausgeber: Der Rektor

Impasse Points: Examples and Counterexamples

by Dipl.-Ing. Gunther Reißig

Lehrstuhl für Theoretische Elektrotechnik
Prof. Dr.-Ing. habil. A. Reibiger

Fakultät Elektrotechnik

ET-IEE-1-1993
1. Nachdruck (3/29/94)

Abstract

In this paper we define classes of impasse points, an important phenomenon of Differential-Algebraic Equations (DAEs). Focussing on DAEs occuring in the analysis of electrical networks, a brief discussion of properties of these classes and examples are given. Some of these examples are counterexamples to published assertions on impasse points.

1 Basic Definitions

1.1 Definition (DAE): *Let $m \in \mathbb{N}$, $n \in \mathbb{N}_+$, $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be open sets and $T \subseteq \mathbb{R}$ an open interval. Let $f: T \times X \times Y \to \mathbb{R}^n \in C^0$ and $g: T \times X \times Y \to \mathbb{R}^m \in C^0$. The system*

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t)) \\ 0 &= g(t, x(t), y(t)) \end{aligned} \tag{1}$$

is called a Differential-Algebraic Equation (DAE). [1] □

1.2 Definition (ADAE): *Let $m \in \mathbb{N}$, $n \in \mathbb{N}_+$, $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be open sets and $f: X \times Y \to \mathbb{R}^n \in C^0$ and $g: X \times Y \to \mathbb{R}^m \in C^0$. The system*

$$\begin{aligned} \dot{x} &= f(x, y) \\ 0 &= g(x, y) \end{aligned} \tag{2}$$

is called an autonomous Differential-Algebraic Equation (ADAE). [1] □

The DAE's to be considered here are sometimes called *semi-explicit DAE's*.

1.3 Definition (Solution): *Let $\varphi: I \to X \times Y \in C^0$, $(t_0, x_0, y_0) \in T \times X \times Y$. φ is said to be a* solution of (1) *(resp.* (2)*)* $:\iff$

(i) $I \subseteq T$ (resp. $I \subseteq \mathbb{R}$) is open and connected.

(ii) $pr_x \circ \varphi \in C^1$

(iii)
$$\begin{array}{rcl} \forall_{t \in I}\, D(pr_x \circ \varphi)(t) &=& f(t, \varphi(t)) \;\wedge\; 0 = g(t, \varphi(t)) \\ (resp.\ 0 \in I \wedge \forall_{t \in I}\, D(pr_x \circ \varphi)(t) &=& f(\varphi(t)) \;\wedge\; 0 = g(\varphi(t))\ ) \end{array}$$

where $pr_x: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes orthogonal projection.

φ is called a C^1-solution of (1) *(resp.* (2)*)* $:\iff$ *φ is a solution of (1) (resp.* (2)*) and $\varphi \in C^1$.*

The solution set S of (1) *(resp.* of (2), C^1-solution set S_{C^1} of (1), C^1-solution set of (2)*) is defined by*

$$\begin{array}{rrl} & S &:= \{\psi | \psi \text{ is a solution of } (1)\} \\ (resp. & S &:= \{\psi | \psi \text{ is a solution of } (2)\} \\ & S_{C^1} &:= \{\psi | \psi \text{ is a } C^1-\text{solution of } (1)\} \\ & S_{C^1} &:= \{\psi | \psi \text{ is a } C^1-\text{solution of } (2)\}) \end{array}$$

The solution set *(resp.* C^1-solution set*)* of (1) according to the initial condition (t_0, x_0, y_0) *($S_{(t_0,x_0,y_0)}$, resp. $S_{C^1,(t_0,x_0,y_0)}$) is defined by*

$$S_{(t_0,x_0,y_0)} := \{\psi \in S | \psi(t_0) = (x_0, y_0)\} \quad (resp. \quad S_{C^1,(t_0,x_0,y_0)} := \{\psi \in S_{C^1} | \psi(t_0) = (x_0, y_0)\})$$

[1] In case $m = 0$ we identify (1) and (2) with ODE's $\dot{x}(t) = f(t, x(t))$ and $\dot{x} = f(x)$, respectively.

The solution set *(resp.* C^1-solution set*)* of (2) according to the initial condition (x_0, y_0) *(*$S_{(x_0,y_0)}$*, resp.* $S_{C^1,(x_0,y_0)}$*) is defined by*

$$S_{(x_0,y_0)} := \{\psi \in S | \psi(0) = (x_0, y_0)\} \quad (resp. \quad S_{C^1,(x_0,y_0)} := \{\psi \in S_{C^1} | \psi(0) = (x_0, y_0)\})$$

φ *is called a* solution *(resp.* C^1-solution*)* of (1) (resp. (2)) passing through (x_0, y_0) at t_0 *(resp.* at 0*)* $:\Longleftrightarrow$

$$\varphi \in S_{(t_0,x_0,y_0)} \quad (resp. \quad \varphi \in S_{C^1,(t_0,x_0,y_0)},\ \varphi \in S_{(x_0,y_0)},\ \varphi \in S_{C^1,(x_0,y_0)})$$

□

As in [3], C^1-solutions are sometimes called *classical solutions*.
One could define a kind of solution without $pr_y \circ \varphi$ being continous. But such solutions do not seem to make too much sense in engineering (see [1, system (49)]). (2)

1.4 Definition (Consistent Initial Value, State Set): *Consider DAE (1) and ADAE (2) and let S and $\tilde{S}$ be the solution sets of (1) and (2). The sets*

$$\begin{array}{lclclcl} & P & := & \{z \in T \times X \times Y | S_z \neq \varnothing\} & \textit{and} & \tilde{P} & := & \{z \in X \times Y | \tilde{S}_z \neq \varnothing\} \\ (resp. & P_{C^1} & := & \{z \in T \times X \times Y | S_{C^1,z} \neq \varnothing\} & \textit{and} & \tilde{P}_{C^1} & := & \{z \in X \times Y | \tilde{S}_{C^1,z} \neq \varnothing\}) \end{array}$$

are called state sets *(resp.* C^1-state sets*)* of (1) *and* (2)*.*
Elements of P and $\tilde{P}$ (resp. P_{C^1} and $\tilde{P}_{C^1}$) are said to be consistent initial values *(resp.* C^1-consistent initial values*)* of (1) *and* (2)*.* □

1.5 Remark: (i) In the following, we use S, S_{C^1}, P, P_{C^1} etc. for both, DAE (1) and ADAE (2).

(ii) $P_{C^1} \subseteq P \subseteq g^{-1}(0)$, $S_{C^1} \subseteq S$, and $\forall_{z \in T \times X \times Y}\, S_{C^1,z} \subseteq S_z$ (resp. $\forall_{z \in X \times Y}\, S_{C^1,z} \subseteq S_z$).

(iii) $P = \cup_{\psi \in S} \operatorname{im}\langle \operatorname{id}|_{\operatorname{dom}\psi}, \psi\rangle$ and $P_{C^1} = \cup_{\psi \in S_{C^1}} \operatorname{im}\langle \operatorname{id}|_{\operatorname{dom}\psi}, \psi\rangle$ in case of DAE (1),
$P = \cup_{\psi \in S} \operatorname{im}\psi = \{\psi(0) | \psi \in S\}$ and $P_{C^1} = \cup_{\psi \in S_{C^1}} \operatorname{im}\psi = \{\psi(0) | \psi \in S_{C^1}\}$ in case of ADAE (2).(3) □

2 Impasse Points

2.1 Definition (Impasse Point): *Consider DAE (1) (resp. ADAE (2)) and let $Q := T \times X \times Y$, $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, and $p = (t_0, x_0, y_0)$ (resp. $Q := X \times Y$ and $p = (x_0, y_0)$). Let further S, S_{C^1}, P, P_{C^1} the sets defined in section 1. p is called*

impasse point *(resp.* C^1-impasse point*)* of the 1st kind *(IP-1, resp. C^1-IP-1)* $:\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \quad (resp. \quad p \in g^{-1}(0) \setminus P_{C^1})$$

impasse point *(resp.* C^1-impasse point*)* of the 2nd kind *(IP-2, resp. C^1-IP-2)* $:\Longleftrightarrow$

$$p \in (\overline{P} \cap Q) \setminus P \quad (resp. \quad p \in (\overline{P_{C^1}} \cap Q) \setminus P_{C^1}) \quad {}^{(4)}$$

---

(2) We do not deal with "Jump Behaviour" here. If we did, we had to consider solutions which are not even continous.
(3) dom: domain of a mapping, im: image of a mapping.
(4) Here, $\overline{P}$ denotes the closure of P in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ (resp. $\mathbb{R}^n \times \mathbb{R}^m$) endowed with the topology induced by some norm. If P^a is the closure of P in the space Q with the topology induced by $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ (resp. $\mathbb{R}^n \times \mathbb{R}^m$), one gets $\overline{P} \cap Q = P^a$ etc.

forward impasse point *(resp.* C^1-forward impasse point*)* of DAE (1) *(FIP, resp. C^1-FIP)* $:\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \quad \wedge \quad \exists_{\psi \in S} \quad (\sup \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$
$$\textit{resp.} \quad p \in g^{-1}(0) \setminus P_{C^1} \quad \wedge \quad \exists_{\psi \in S_{C^1}} (\sup \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

forward impasse point *(resp.* C^1-forward impasse point*)* of ADAE (2) *(FIP, resp. C^1-FIP)* $:\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \quad \wedge \quad \exists_{\psi \in S} \quad (\sup \operatorname{dom} \psi \in \mathbb{R} \wedge \lim_{t \to \sup \operatorname{dom} \psi} \psi(t) = (x_0, y_0))$$
$$\textit{resp.} \quad p \in g^{-1}(0) \setminus P_{C^1} \quad \wedge \quad \exists_{\psi \in S_{C^1}} (\sup \operatorname{dom} \psi \in \mathbb{R} \wedge \lim_{t \to \sup \operatorname{dom} \psi} \psi(t) = (x_0, y_0))$$

backward impasse point *(resp.* C^1-backward impasse point*)* of DAE (1) *(BIP, resp. C^1-BIP)* $:\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \quad \wedge \quad \exists_{\psi \in S} \quad (\inf \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$
$$\textit{resp.} \quad p \in g^{-1}(0) \setminus P_{C^1} \quad \wedge \quad \exists_{\psi \in S_{C^1}} (\inf \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

backward impasse point *(resp.* C^1-backward impasse point*)* of ADAE (2) *(BIP, resp. C^1-BIP)* $:\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \quad \wedge \quad \exists_{\psi \in S} \quad (\inf \operatorname{dom} \psi \in \mathbb{R} \wedge \lim_{t \to \inf \operatorname{dom} \psi} \psi(t) = (x_0, y_0))$$
$$\textit{resp.} \quad p \in g^{-1}(0) \setminus P_{C^1} \quad \wedge \quad \exists_{\psi \in S_{C^1}} (\inf \operatorname{dom} \psi \in \mathbb{R} \wedge \lim_{t \to \inf \operatorname{dom} \psi} \psi(t) = (x_0, y_0))$$

□

In the sequel, the sets of impasse points of the first kind, of the second kind, of forward impasse points, and of backward impasse points (resp. the corresponding kinds of C^1-impasse points) will be refered to by I_1, I_2, I_F, and I_B (resp. $I_{C^1,1}$, $I_{C^1,2}$, $I_{C^1,F}$, and $I_{C^1,B}$).

2.2 Lemma: *Consider DAE (1) or ADAE (2). Then*

(i) $I_B \cup I_F \subseteq I_2 \subseteq I_1$

(ii) $I_{C^1,B} \cup I_{C^1,F} \subseteq I_{C^1,2} \subseteq I_{C^1,1}$

(iii) $I_1 \subseteq I_{C^1}$ □

Proof:

(i) We show (i) only for DAE (1):
Let $p = (t_0, x_0, y_0) \in I_B$, then $p \in g^{-1}(0) \setminus P$ and $\exists_{\psi \in S} \sup \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0)$, i.e. $\lim_{t \to t_0}(t, \psi(t)) = p$. Since $\operatorname{im}\langle \operatorname{id}|_{\operatorname{dom}\psi}, \psi \rangle \subseteq P$ (Remark 1.5), we get $p \in \overline{P}$, e.g. $p \in g^{-1}(0)$, $p \notin P$, and $p \in \overline{P} \implies p \in (\overline{P} \cap g^{-1}(0)) \setminus P \subseteq (\overline{P} \cap T \times X \times Y) \setminus P = I_2$.
Let $p \in I_2 = (\overline{P} \cap T \times X \times Y) \setminus P$. Since $g^{-1}(0)$ is closed (in $T \times X \times Y$) and $P \subseteq g^{-1}(0)$, we get $\overline{P} \cap T \times X \times Y \subseteq g^{-1}(0) \Longrightarrow p \in g^{-1}(0) \setminus P = I_1$.

(ii) Equivalent to (i).

(iii) $P_{C^1} \subseteq P \subseteq g^{-1}(0) \Longrightarrow I_1 = g^{-1}(0) \setminus P \subseteq g^{-1}(0) \setminus P_{C^1} = I_{C^1,1}$. □

2.3 Remark: (i) As we will see in this section, the kinds of impasse points defined above are in general different from each other. We will show, at least, that $I_1 \neq I_{C^1,1}$ (Example 2.9), $I_{C^1,1} \neq I_{C^1,2}$ and $I_1 \neq I_2$ (Example 2.10), $I_2 \neq I_B \cup I_F$ and $I_{C^1,2} \neq I_{C^1,B} \cup I_{C^1,F}$ (Example 2.11).

(ii) One may think about additional assumptions to prove $I_2 \subseteq I_{C^1,2}$, $I_B \subseteq I_{C^1,B}$, and $I_F \subseteq I_{C^1,F}$.

(iii) Another kind of impasse point would be the definition of a BIP with $p \notin P$ substituted by the weaker requirement that ψ is not extendable beyond p. $\square$

Considering DAE (1) (resp. ADAE (2)) in case $m = 0$, we have, in fact, an ODE:

$$\dot{x}(t) = f(t, x(t)) \tag{3}$$

resp.

$$\dot{x} = f(x) \tag{4}$$

As is well known from the theory of ODE's, we get $P = P_{C^1} = T \times X$ and $P = P_{C^1} = X$ for systems (3) and (4), respectively. Thus, impasse points cannot occur in case of ODE's. We consider that to be an important difference between DAE's and ODE's. However, there are DAE's being equivalent to ODE's (for a definition of *equivalence* see [4]).

2.4 Lemma: *Consider DAE (1) (resp. ADAE (2)) and let* $p = (t_0, x_0, y_0) \in T \times X \times Y$, $\tilde{p} = (t_0, x_0)$, $pr: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^n$ *the orthogonal projection, and* $D_y g = D_3 g$
(resp. $p = (x_0, y_0) \in X \times Y$, $\tilde{p} = x_0$, $pr: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, *and* $D_y g = D_2 g$*).*
Then

(i) $\exists_{U \in \mathcal{U}(\tilde{p})} \exists_{h: U \to Y \in C^0}\, h(\tilde{p}) = y_0 \wedge \forall_{u \in U}\, g(u, h(u)) = 0 \Longrightarrow p \in P \quad \wedge \tilde{p} \in (pr(P))^0$ [(5)(6)]

(ii) $\exists_{U \in \mathcal{U}(\tilde{p})} \exists_{h: U \to Y \in C^1}\, h(\tilde{p}) = y_0 \wedge \forall_{u \in U}\, g(u, h(u)) = 0 \Longrightarrow p \in P_{C^1} \wedge \tilde{p} \in (pr(P_{C^1}))^0$

(iii) $\exists_{V \in \mathcal{U}(p)}\, g|_V \in C^1 \wedge g(p) = 0 \wedge D_y g(p)$ *bijective* $\Longrightarrow p \in P_{C^1} \wedge \tilde{p} \in (pr(P_{C^1}))^0$ $\square$

Proof:
We show (i) - (iii) only for DAE (1):

(i) Without loss of generality, we assume that $U = J \times Q$, where J is an open interval and $Q \subseteq X$ is open.
Consider the ODE $\dot{x}(t) = \tilde{f}(t, x(t))$, where $\tilde{f}: J \times Q \to \mathbb{R}^n: (t, c) \mapsto f(t, c, h(t, c)) \in C^0$. Obviously, there exists a solution $\varphi: I \to Q \in C^1$ with $\varphi(t_0) = x_0$ ($I \subseteq J$ is an open interval).Set $\psi: I \to Q \times Y: t \mapsto (\varphi(t), h(t, \varphi(t)))$. Since $I \times pr_x(\operatorname{im} \psi) \subseteq J \times Q \subseteq U \Longrightarrow$ $\forall_{t \in I}\, g(t, \varphi(t), h(t, \varphi(t)) = g(t, \psi(t)) = 0$.
Further, $\psi \in C^0$, $\varphi = pr_x \circ \psi \in C^1$, and

$$D(pr_x \circ \psi)(t) = \dot{\varphi}(t) = \tilde{f}(t, \varphi(t)) = f(t, \varphi(t), h(t, \varphi(t)) = f(t, \psi(t))$$

for all $t \in I$.
Further, $\varphi(t_0) = x_0$ and $h(t_0, x_0) = y_0 \Longrightarrow \psi(t_0) = (x_0, y_0) \Longrightarrow \psi \in S_p \Longrightarrow p \in P$.
Let $(t_1, x_1) \in U$. Then $(t_1, x_1, h(t_1, x_1))$ meets the requirements of (i). Thus,
$(t_1, x_1, h(t_1, x_1)) \in P \Longrightarrow (t_1, x_1) \in pr(P) \Longrightarrow (t_0, x_0) \in (pr(P))^0$

(ii) Equivalent to (i).

(iii) Applying the Implicit Function Theorem [2, 10.2.1] we get

$$\exists_{W \in \mathcal{U}((t_0, x_0))} \exists!_{h: W \to Y \in C^1} h(t_0, x_0) = y_0 \wedge \forall_{w \in W} (w, h(w)) \in V \wedge g(w, h(w)) = 0$$

By (ii) we get the assertion. $\square$

(5) $((pr(P))^0$ denotes the interior of $pr(P)$ in the space $T \times X$ (resp. X).
(6) $\mathcal{U}(q)$ is the set of open neighbourhoods of q in the space considered (here: $T \times X$ and X, respectively).

2.5 Remark: Open questions are:

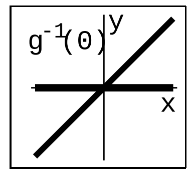
(i) Is it possible to prove "$\Longleftarrow$"-parts of Lemma 2.4, at least under stronger assumptions ?

(ii) What are sufficient assumptions for uniqueness of solutions passing through (x_0, y_0) at t_0 (resp. at 0) ?

(iii) What are sufficient assumptions to obtain $p \in P^{00}_{C^1}$ (where $(.)^{00}$ denotes the interior in the space $g^{-1}(0)$) instead of $\tilde{p} \in (pr(P_{C^1}))^0$? $\square$

In [1, Lemma 1 and 2] one can find statements similar to those of Lemma 2.4. Concerning (i) and (ii), Lemma 2.4 is more general since h does not need to be unique:

2.6 Example:



$$\begin{aligned} \dot{x} = \quad & f(x,y) \\ 0 = \quad & g(x,y) \quad = \ y\,(y-x) \end{aligned}$$

Let $g(x_0, y_0) = 0$ and let $h\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \left\{ \begin{smallmatrix} 0, & if \ \ y_0=0 \\ x, & else \end{smallmatrix} \right\}$. Obviously, using $U = \mathbb{R}$ as a neighbourhood of x_0, one gets $h \in C^1$, $h(x_0) = y_0$, $g(x, h(x)) = \left\{ \begin{smallmatrix} g(x,0), & if \ \ y_0=0 \\ g(x,x), & else \end{smallmatrix} \right\} = 0$, and by Lemma 2.4.ii, $(x_0, y_0) \in P_{C^1}$, e.g. $P_{C^1} = g^{-1}(0)$.
If $(x_0, y_0) = (0, 0)$, Lemma 2.4 remains applicable although no *unique* h exists. $\square$

2.7 Lemma: *Consider DAE (1) (resp. ADAE (2)) and assume $g \in C^1$. Let $p \in T \times X \times Y$ and let $pr_{t,x}\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^n$ be the orthogonal projection (resp. $p \in X \times Y$, $pr_x\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection). Then*

$$\begin{aligned} & p \in g^{-1}(0) \ \wedge \quad (1, f(p)) \notin pr_{t,x}(\ker(Dg(p))) \quad \Longrightarrow p \in I_{C^1,1} \\ (resp. \quad & p \in g^{-1}(0) \ \wedge \qquad\quad f(p) \ \ \notin pr_x \ \ (\ker(Dg(p))) \quad \Longrightarrow p \in I_{C^1,1} \end{aligned}$$

$\square$

Proof: The proof is done for DAE (1) only:
Let $p = (t_0, x_0, y_0)$ and assume $p \notin I_{C^1,1}$. Then, there exists a solution $\varphi \in S_{C^1,p}$ of DAE (1). Let be $\psi\colon \operatorname{dom}\varphi \to T \times X \times Y\colon t \mapsto (t, \varphi(t))$. Obviously, $\psi(t_0) = p$, $g \circ \psi = 0$, and $D(g \circ \psi) = 0$. Thus,

$$\begin{aligned} 0 \quad &= \quad D(g \circ \psi)(t_0) = Dg(\psi(t_0)) \circ D\psi(t_0) = Dg(p) \circ D\langle \mathrm{id}\,|_{\operatorname{dom}\varphi}, \varphi\rangle(t_0) \\ &= \quad Dg(p) \circ \langle \mathrm{id}\,|_{\mathbb{R}}, f(p), D(pr_y \circ \psi)(t_0)\rangle \end{aligned}$$

$\Longrightarrow (1, f(p)) \in pr_{t,x}(\ker(Dg(p)))$ [(7)]. Contradiction. $\square$

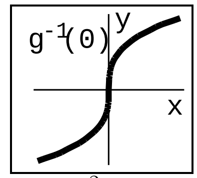
2.8 Remark: (i) If $g^{-1}(0)$ of ADAE (2) is a manifold and $Dg(x_0, y_0)$ has full rank, Lemma 2.7 is equivalent to the argument that for any C^1-solution φ passing through (x_0, y_0) at 0, $\dot{\varphi}(0)$ has to lie in the tangent space of $g^{-1}(0)$ at (x_0, y_0).[4]
An analogous argument holds for DAE (1).

(ii) Under the assumptions of Lemma 2.7, p does not need to be in I_1 (Example 2.9), nor in $I_{C^1,2}$ (Example 2.10).

(7) If $\operatorname{dom} f_1 = \cdots = \operatorname{dom} f_k$, $\langle f_1, .., f_k\rangle$ is defined as $\langle f_1, .., f_k\rangle\colon \operatorname{dom} f_1 \to \operatorname{im} f_1 \times .. \times \operatorname{im} f_k\colon t \mapsto (f_1(t), .., f_k(t))$

(iii) Considering ADAE (2) and following [1], let $S_3 := \{p \in g^{-1}(0) | f(p) \notin pr_x(\ker(Dg(p)))\}$ and let I be the set of impasse points as defined in [1]. Now, Lemma 2.7 means $S_3 \subseteq I_{C^1,1}$.
In [1, Lemma 3], Chua and Deng assert that $I \subseteq S_3$. That can shown to be wrong even in case $g^{-1}(0)$ is a smooth manifold. By Example 2.12 we obtain that even BIP's and FIP's do not need to lie in S_3.
If [1, Lemma 3] was right, there would not exist any DAE's with index greater than 1 (for a definition of *index* see [4]). □

2.9 Example:



$$\begin{aligned} \dot{x} = \quad f(x,y) \quad &= 1 \\ 0 = \quad g(x,y) \quad &= y^3 - x \end{aligned}$$

By Lemma 2.4.iii one obtains $g^{-1}(0) \setminus \{(0,0)\} \subseteq P_{C^1}$, because $D_y g(x,y) = (3y^2)$ is bijective for all points (x,y) of $g^{-1}(0) \setminus \{(0,0)\}$.
To tackle the point $(0,0)$, we apply Lemma 2.4.i with $U = \mathbb{R}$ and $h(x) = |x|^{1/3} \operatorname{sign}(x)$. Thus, $(0,0) \in P$.
Further, $(0,0) \in I_{C^1,1}$, because $(0,0)$ meets the requirements of Lemma 2.7:

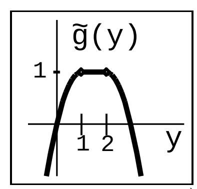
$$\ker(Dg(0,0)) = \ker((-1,0)) = \{0\} \times \mathbb{R} \quad \text{and} \quad f(0,0) = 1 \notin \{0\} = pr_x(\ker(Dg(0,0)))$$

After all we have:

$$\begin{aligned} P_{C^1} &= g^{-1}(0) \setminus \{(0,0)\}, & P &= g^{-1}(0), \\ I_{C^1,1} &= \{(0,0)\}, & I_1 &= I_2 = I_B = I_F = \emptyset \end{aligned}$$

and by further investigations: $I_{C^1,2} = I_{C^1,B} = I_{C^1,F} = \{(0,0)\}$. □

2.10 Example:



$$\begin{aligned} \dot{x} = \quad f(x,y) \quad &= 1 \\ 0 = \quad g(x,y) \quad &= x - \tilde{g}(y) \end{aligned}$$

with $\tilde{g} \in C^\infty$, $\tilde{g}|_{[1,2]} = 1$, and $\tilde{g}|_{\mathbb{R}\setminus[1,2]} < 1$. ($\tilde{g}$ exists, e.g. constructed by regularisation.)
Consider $p \in \{1\} \times]1,2[$[8] and apply Lemma 2.7:

$$\ker(Dg(p)) = \ker((1,0)) = \{0\} \times \mathbb{R} \quad \text{and} \quad 1 \notin \{0\} = pr_x(\ker(Dg(p)))$$

$\Longrightarrow \{1\} \times]1,2[\cap P_{C^1} = \emptyset \Longrightarrow \{1\} \times]1,2[\cap I_{C^1,2} = \emptyset$.
At this time, wo do not have an appropriate theorem to show that

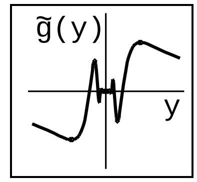
$$\{1\} \times]1,2[\subseteq I_1 \quad \text{and} \quad \{1\} \times]1,2[\cap I_2 = \emptyset$$

but that seems to be obvious here. □

---

[8] $]a,b[$ denotes the open interval, $[a,b]$ the closed etc.

2.11 Example:



$$\begin{aligned} \dot{x} = \quad & f(x,y) & = \; & 1 \\ 0 = \quad & g(x,y) & = \; & x - \tilde{g}(y) \end{aligned}$$

with $\tilde{g}(y) = \exp(-\frac{1}{y^2})\sin(\frac{2\pi}{y})$ and $\tilde{g} \in C^\infty$ (see Lemma A.1, p. 11). (9)

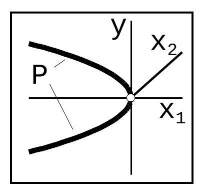
1. $(0,0) \in \overline{P_{C^1}}$**:** Let $(x,y) \in g^{-1}(0) \setminus \{(0,0)\}$, then $\tilde{g}'(y) = \exp(-\frac{1}{y^2})(\frac{2}{y^3}\sin(\frac{2\pi}{y}) - \frac{2\pi}{y^2}\cos(\frac{2\pi}{y}))$. Let $\mathcal{Z} = \tilde{g}^{-1}(0) \setminus \{0\} = \{\frac{2}{k} | k \in \mathbb{Z} \setminus \{0\}\}$. Obviously, $\forall_{y \in \mathcal{Z}}\, \tilde{g}'(y) \neq 0$, and by Lemma 2.4.iii we obtain $\{0\} \times \mathcal{Z} \subseteq P_{C^1}$ and since $\lim_{k\to\infty}(0, \frac{2}{k}) = (0,0)$ we get $(0,0) \in \overline{P_{C^1}}$.

2. $(0,0) \notin I_B \cup I_F \cup I_{C^1,B} \cup I_{C^1,F}$**:** Assume $\exists_{\varphi \in S}(\sup \operatorname{dom}\varphi \in \mathbb{R} \wedge \lim_{t \to \sup \operatorname{dom}\varphi} \varphi(t) = (0,0))$. Without loss of generality, we can further assume that $\operatorname{dom}\varphi =]-\varepsilon, \varepsilon[$. Let $\varphi = \langle \varphi_x, \varphi_y \rangle \in C^0$ and $\varphi_x \in C^1$. Considering φ_y we have $\lim_{t\to\varepsilon} \varphi_y(t) = 0$.
Case 1: $\varphi_y = 0$
$\Longrightarrow \operatorname{im}\varphi_x \subseteq \tilde{g}^{-1}(0) = \mathcal{Z} \cup \{0\}$. Since $\dot{\varphi}_x = 1$, $\operatorname{im}\varphi_x$ contains an interval. Contradiction.
Case 2: $\exists_{t_0 \in]-\varepsilon,\varepsilon[}\, \varphi(t_0) \neq 0$. (Without loss of generality, let $\varphi_y(t_0) > 0$.)
$\varphi_y \in C^0 \wedge \lim_{t\to\varepsilon} \varphi_y(t) = 0 \Longrightarrow]0, \varphi_y(t_0)] \subseteq \operatorname{im}\varphi_y$. Obviously, $\exists_{k \in \mathbb{N}_+} \frac{2}{k+1}, \frac{2}{k} \in]0, \varphi_y(t_0)[\subseteq \operatorname{im}\varphi_y$. Let $t_1, t_2 \in \operatorname{dom}\varphi$, $t_1 < t_2$, with $\varphi_y(t_1) = \frac{2}{k+1}$ and $\varphi_y(t_2) = \frac{2}{k}$. $\Longrightarrow$
$\exists_{t \in]t_1,t_2[}\, \dot{\varphi}_x(t)(t_2 - t_1) = \varphi_x(t_2) - \varphi_x(t_1) = 0$. Contradiction.

3. $(0,0) \notin P$**:** Assume $\exists_{\varphi \in S_{(0,0)}}$ and $]-\varepsilon,\varepsilon[\subseteq \operatorname{dom}\varphi$. Set $\psi\colon]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[\to \mathbb{R}^2\colon t \mapsto \varphi(t - \frac{\varepsilon}{2}) \Longrightarrow$ $\lim_{t\to\frac{\varepsilon}{2}} \psi(t) = \lim_{t\to 0}\varphi(t) = 0$, which contradicts 2.

So, it follows: $(0,0)$ is an IP-2, a C^1-IP-2 but is neither a BIP nor a C^1-BIP nor a FIP nor a C^1-FIP.
□

2.12 Example:



$$\begin{aligned} \dot{x}_1 = \quad & f_1(x_1,x_2,y) & = 1 \\ \dot{x}_2 = \quad & f_2(x_1,x_2,y) & = x_1 + y^2 \\ 0 = \quad & g(x_1,x_2,y) & = x_2 \end{aligned}$$

where $f = \langle f_1, f_2 \rangle$. Obviously, for all $(x_{1,0}, y_0) \in \mathbb{R}^2$ satisfying $x_{1,0} < 0$ and $x_{1,0} + y_0^2 = 0$,

$$\varphi\colon]x_{1,0}, -x_{1,0}[\to \mathbb{R}^2 \times \mathbb{R}\colon t \mapsto \left(x_{1,0} + t, 0, \sqrt{-x_{1,0} - t} \begin{Bmatrix} 1 & \text{if } y_0 > 0 \\ -1 & \text{else} \end{Bmatrix} \right)$$

is in S_{C^1} and hence, $\{(x_1, 0, y) \in \mathbb{R}^2 \times \mathbb{R} | x_1 < 0 \wedge x_1 + y^2 = 0\} \subseteq P_{C^1}$.
Let $\langle x_1, x_2, y \rangle$ be a solution. Then $x_2 = 0$ and, by that fact, $0 = \dot{x}_2 = x_1 + y^2$, and hence, $P_{C^1} \subseteq \{(x_1, 0, y) | x_1 + y^2 = 0\}$.
Assume $\langle x_1, x_2, y \rangle \in S_{(0,0,0)}$, then $\exists_{\varepsilon > 0}]-\varepsilon, \varepsilon[\subseteq \operatorname{im} x_1$, because $\dot{x}_1(0) = 1$. Contradiction.
Thus, $P = P_{C^1} = \{(x_1, 0, y) | x_1 + y^2 = 0 \wedge x_1 < 0\}$.(10)
We have seen that $(0,0,0)$ is a FIP and a C^1-FIP. Now, as in [1] (see also Remark 2.8.iii), let us consider Dg:

$$Dg = (0,1,0) \Longrightarrow \ker Dg = \mathbb{R} \times \{0\} \times \mathbb{R} \Longrightarrow pr_x \ker Dg = \mathbb{R} \times \{0\}$$

Since $f(0,0,0) = (1,0) \in pr_x \ker Dg(0,0,0)$ we have $(0,0,0) \notin S_3$. That contradicts [1, Lemma 3]. Note that Dg has full rank everywhere and $P = P_{C^1}$ is a smooth manifold. □

(9) In this example we mean by a function, unless its domain is explicitly given, its continous extension to $\mathbb{R}$.
(10) We could use here the procedure given in [4] to get P_{C^1}, but want to get P as well.

3 Limit Points

In this section we follow [1] as far as we try to follow the considerations which have lead to [1, Theorem 1] and therefore, deal with autonomous DAE's with $Y = \mathbb{R}^m$ only.

3.1 Definition: *Let* $h\colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \in C^0$, $(\lambda_0, y_0) \in h^{-1}(0)$.
(λ_0, y_0) *is called* right limit point of $h^{-1}(0)$ $:\Longleftrightarrow$

$$\exists_{N \in \mathcal{U}((\lambda_0, y_0))}\, N \cap h^{-1}(0) \cap ((\lambda_0 + \mathbb{R}_+) \times \mathbb{R}^m) = \varnothing^{(11)}$$

(λ_0, y_0) *is called* left limit point of $h^{-1}(0)$ $:\Longleftrightarrow$

$$\exists_{N \in \mathcal{U}((\lambda_0, y_0))}\, N \cap h^{-1}(0) \cap ((\lambda_0 - \mathbb{R}_+) \times \mathbb{R}^m) = \varnothing$$

□

The definition of a limit point given is equivalent to that given in [1] (see Lemma A.2).

3.2 Definition: *Consider ADAE (2) with* $Y = \mathbb{R}^m$ *and let* $(x_0, y_0) \in g^{-1}(0)$. *Then*

$$h\colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m\colon (\lambda, y) \mapsto g(x_0 + \lambda f(x_0, y_0), y)$$

is called cut mapping of ADAE (2) at (x_0, y_0). □

The purpose of [1, section 2.2] is to decide wether a point $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ of ADAE (2) is an impasse point or not by solving a static bifurcation problem. Namely, [1, Theorem 1] asserts that

(T1) Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, let h be the cut mapping of ADAE (2) at (x_0, y_0), and let $Y = \mathbb{R}^m$. Then

(x_0, y_0) is a FIP (resp. BIP) of ADAE (2) $\Longleftrightarrow$ $(x_0, y_0) \in g^{-1}(0) \wedge (0, y_0)$ is a right (resp. left) limit point of $h^{-1}(0)$

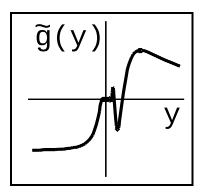
The proof of (T1) in [1] is wrong, and so do both parts (" $\Longrightarrow$ " and " $\Longleftarrow$ ") of (T1) itself. The falsity of the proof is, from our point of view, mainly due to the fact that there are two conjectures (C1,C2) used within it, both of which can be shown to be wrong (Examples 3.3, 3.5, and 3.6).
One idea of the proof mentioned is:

(C1) Consider ADAE (2) with $Y = \mathbb{R}^m$, $n = 1$, $f = 1$, and $(0, y_0) \in g^{-1}(0)$. Then

$(0, y_0)$ is a FIP (resp. BIP) of ADAE (2) $\Longleftrightarrow$ $(0, y_0)$ is a right (resp. left) limit point of $g^{-1}(0)$.

3.3 Example:

$$\begin{aligned} \dot{x} = \quad & f(x, y) & = 1 \\ 0 = \quad & g(x, y) & = x - \tilde{g}(y) \end{aligned}$$



where $\tilde{g}(y) = \exp(-\frac{1}{y^2}) \begin{Bmatrix} \sin(\frac{2\pi}{y}) & y > 0 \\ -1 & \text{else} \end{Bmatrix}$ with $\tilde{g} \in C^\infty$ (see Lemma A.1, p. 11). Then

$$\forall_{x_0 \in]-1,0[}\, \varphi\colon]x_0, -x_0[\to \mathbb{R}^2\colon t \mapsto \left(x_0 + t, -\sqrt{\frac{-1}{\ln(-(x_0 + t))}} \right) \in S_{C^1} \wedge \lim_{t \to -x_0} \varphi(t) = (0, 0)$$

As in Example 2.11, it is easy to show that $(0, 0) \notin P \Longrightarrow (0, 0)$ is a FIP and a C[1]-FIP. But $(0, 0)$ is neither a left nor a right limit point of $g^{-1}(0)$. Here, g is itself the cut mapping at $(0, 0)$ and thus, this example contradicts (C1) as well as the " $\Longrightarrow$ "-part of (T1), i.e. [1, Theorem 1]. □

(11) $0 \notin \mathbb{R}_+ \cup \mathbb{R}_-$

3.4 Definition: *Consider ADAE (2) with $Y = \mathbb{R}^m$ and let $(x_0, y_0) \in g^{-1}(0)$. Then the system*

$$\begin{aligned} \dot{\lambda} &= 1 \in \mathbb{R} \\ 0 &= h(\lambda, y) \end{aligned} \tag{5}$$

with $\operatorname{dom} h = \mathbb{R} \times \mathbb{R}^m$ *and* $h(\lambda, y) = g(x_0 + \lambda f(x_0, y_0), y)$ *is called* first order ADAE corresponding to ADAE (2) at (x_0, y_0). □

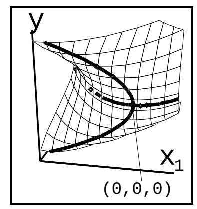
Using the definition given above, we can formulate another idea of the proof in question as follows:

(C2) Consider ADAE (2), let $(x_0, y_0) \in g^{-1}(0)$ and let ADAE (5) be the first order ADAE corresponding to (2) at (x_0, y_0). Then

$$(x_0, y_0) \text{ is a FIP (resp. BIP) of (2)} \iff (0, y_0) \text{ is a FIP (resp. BIP) of (5)}$$

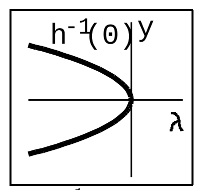
3.5 Example:

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= 2x_1 \\ 0 &= x_2 - (x_1 + y^2)^2 = g(x_1, x_2, y) \end{aligned} \tag{6}$$



Consider the point $(0,0,0)$. Obviously, $\varphi\colon]-1,1[\to \mathbb{R}^3\colon t \mapsto (t, t^2, 0) \in S_{C^1,(0,0,0)} \implies (0,0,0) \in P_{C^1}$.
Consider the corresponding first order ADAE at $(0,0,0)$:

$$\begin{aligned} \dot{\lambda} &= 1 \\ 0 &= -(\lambda + y^2)^2 = h(\lambda, y) \end{aligned} \tag{7}$$



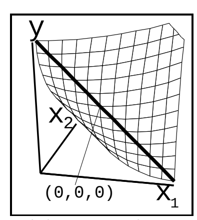
Obviously, $(0,0)$ is an FIP and a C^1-FIP of (7) and a right limit point of $h^{-1}(0)$:

$$h(\lambda, y) = g(0 + \lambda \cdot 1, 0 + \lambda \cdot 2 \cdot 0, y) = -(\lambda + y^2)^2$$

But $(0,0)$ is not even an IP-1 of (6). That contradicts (C2) as well as the " $\Longleftarrow$ "-part of (T1), i.e. [1, Theorem 1]. □

3.6 Example:

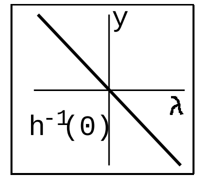
$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= f_2(x_1) \\ 0 &= x_2 - (x_1 + y)^2 = g(x_1, x_2, y) \end{aligned} \tag{8}$$



where $f_2 \in C^\infty$, $f_2|_{\mathbb{R}_- \cup \{0\}} = 0$, and $f_2|_{\mathbb{R}_+} < 0$. (f_2 exists, e.g. constructed by regularisation.)
Assume $\langle x_1, x_2, y \rangle = \psi \in S_{(0,0,0)}$, without loss of generality let $\operatorname{dom} \psi =]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$. Then $\forall_{t \in \operatorname{dom} \psi}\, x_1(t) = t \wedge x_2(t) \geq 0$.
Considering $x_2(\frac{\varepsilon}{2}) \geq 0$ we obtain: $\exists_{t \in]0, \frac{\varepsilon}{2}[}\, \dot{x}_2(t) = \frac{2}{\varepsilon}(x_2(\frac{\varepsilon}{2}) - x_2(0)) = \frac{2}{\varepsilon} x_2(\frac{\varepsilon}{2}) \geq 0$ which contradicts $f_2|_{\mathbb{R}_+} < 0$. $\implies (0,0,0) \notin P$
Consider $\varphi\colon]-1,1[\to \mathbb{R}^3\colon t \mapsto (t-1, 0, -t+1)$. Obviously, $\varphi \in S_{C^1}$ and $\lim_{t \to 1} \varphi(t) = (0,0,0)$. Thus, $(0,0,0)$ is a FIP as well as a C^1-FIP of (8).

Consider the corresponding first order ADAE at $(0,0,0)$:

$$\begin{aligned}\dot{\lambda} &= 1\\ 0 &= -(\lambda+y)^2 = h(\lambda,y)\end{aligned} \tag{9}$$



It is easy to see that $(0,0,0)$ is neither a FIP nor a C^1-FIP of (9) nor a right limit point of $h^{-1}(0)$:

$$h(\lambda,y) = g(0+\lambda\cdot 1, 0+\lambda f_2(0), y) = -(\lambda+y)^2$$

Thus, this example contradicts (C2) as well as the " $\Longrightarrow$ "-part of (T1), i.e. [1, Theorem 1]. □

Note that Examples 3.3, 3.5, and 3.6 do not only meet the requirements of [1, Theorem 1], but also satisfy the additional assumptions of the algorithm given in [1, Part II], namely, $f,g,h \in C^\infty$.

A Appendix

A.1 Lemma: *Let $k \in \mathbb{N}_+$, $\varepsilon \in \mathbb{R}_+$, and $g\colon \mathbb{R}\setminus[-\frac{1}{\varepsilon^k},\frac{1}{\varepsilon^k}] \to \mathbb{R}$ bounded.*
Let further $f\colon]-\varepsilon,\varepsilon[\to \mathbb{R}\colon x \mapsto \left\{\begin{array}{ll} 0 & \text{if } x=0 \\ \exp(-\frac{1}{x^2})g(\frac{1}{x^k}) & \text{else}\end{array}\right\}$. Then

(i) f is differentiable at 0 and $f'(0)=0$.

(ii) Let $r \in \mathbb{N}\cup\{\infty\}$, $g \in C^r$, and $\forall_{i\in\mathbb{N}_+, i\le r}\, g^{(i)}$ be bounded. Then $f \in C^r \wedge \forall_{i\in\mathbb{N}_+, i\le r}\, f^{(i)}(0)=0$. □

Proof:

(i) $\lim_{x\to 0} f(x) = 0$ is obvious. Let $n \in \mathbb{N}$. Then

$$\lim_{x\to 0} x^{-n}\exp(-\frac{1}{x^2}) = \lim_{|x|\to\infty} x^n \exp\left(-x^2\right) = 0 \tag{10}$$

and

$$\lim_{\delta\to 0}\left|\frac{f(\delta)-f(0)}{\delta}\right| = \lim_{\delta\to 0}\left|\frac{f(\delta)}{\delta}\right| \le M \lim_{\delta\to 0}\left|\delta^{-1}\exp(-\frac{1}{\delta^2})\right| = 0$$

(ii.a) Let P the set of polynomials with real coefficients and set $\tilde{f} = f|_{]-\varepsilon,\varepsilon[\setminus\{0\}}$. Then, for all $i \in \{0,\cdots,r\}$, $\tilde{f}^{(i)}$ can be written as follows:

$$\tilde{f}^{(i)}(x) = \exp(-\frac{1}{x^2})\sum_{j=0}^{i} g^{(j)}(\frac{1}{x^k})\frac{p_j(x)}{q_j(x)} \tag{11}$$

where $\forall_{j\in\{0,\cdots,i\}}\, p_j, q_j \in P$.
Proof of (11):
(11) holds for $\tilde{f}^{(0)} = \tilde{f}$. Let $i \in \mathbb{N}_+$, $i < r$ and assume that (11) holds for $\tilde{f}^{(i)}$. Then

$$\begin{aligned}\tilde{f}^{(i+1)}(x) &= 2x^{-3}\exp(-\frac{1}{x^2})\sum_{j=0}^{i} g^{(j)}(\frac{1}{x^k})\frac{p_j(x)}{q_j(x)}\\ &\quad+\exp(-\frac{1}{x^2})\sum_{j=0}^{i}\left(g^{(j+1)}(\frac{1}{x^k})(-k)x^{-(k+1)}\frac{p_j(x)}{q_j(x)} + g^{(j)}(\frac{1}{x^k})\frac{p_j'(x)q_j(x)-p_j(x)q_j'(x)}{(q_j(x))^2}\right)\end{aligned}$$

which can obviously be brought into form (11). Thus, (11) holds for $\tilde{f}^{(i+1)}$.

(ii.b) We show $\forall_{i\in\mathbb{N},i\le r} \lim_{x\to 0} f^{(i)}(x) = 0$:

$$\begin{aligned}\lim_{x\to 0}\left|f^{(i)}(x)\right| = \lim_{x\to 0}\left|\tilde{f}^{(i)}(x)\right| &\le \lim_{x\to 0}\sum_{j=0}^{i} M_j \left|\frac{\exp(-\frac{1}{x^2})}{q_j(x)}\right| \quad (\textit{since } p_j \in C^0)\\ &= \sum_{j=0}^{i} M_j \lim_{x\to 0}\left|\frac{\exp(-\frac{1}{x^2})}{q_j(x)}\right|\end{aligned}$$

For all $j \in \{0,\cdots,i\}$ there exist $n_j \in \mathbb{N}$ and $\tilde{q}_j \in P$ with $\tilde{q}_j(0) \neq 0$ and $q_j(x) = x^{n_j}\tilde{q}_j(x)$ for all $x \in \operatorname{dom}\tilde{f}$. $\Longrightarrow$

$$\begin{aligned}\lim_{x\to 0}\left|f^{(i)}(x)\right| &\le \sum_{j=0}^{i} M_j \lim_{x\to 0}\left|x^{-n_j}\exp(-\frac{1}{x^2})\frac{1}{\tilde{q}_j(x)}\right|\\ &\le \sum_{j=0}^{i} M_j L_j \lim_{x\to 0}\left|x^{-n_j}\exp(-\frac{1}{x^2})\right| \quad (\textit{since } \tilde{q}_j \in C^0 \textit{ and } \tilde{q}_j(0)\neq 0)\\ &= 0 \quad (\textit{see } (10))\end{aligned}$$

(ii.c) We show $\forall_{i\in\mathbb{N},i\le r} f^{(i)}(0) = 0$:

The assertion holds for $i = 0$. Let $i \in \mathbb{N}_+$, $i < r$ and assume $f^{(i)}(0) = 0$. Obviously, $\frac{\tilde{f}^{(i)}(x)}{x}$ is of form (11). It follows as in (ii.b):

$$\lim_{\delta\to 0}\frac{f^{(i)}(\delta) - f^{(i)}(0)}{\delta} = \lim_{\delta\to 0}\frac{f^{(i)}(\delta)}{\delta} = \lim_{\delta\to 0}\frac{\tilde{f}^{(i)}(\delta)}{\delta} = 0$$

$\Longrightarrow f^{(i+1)}(0)$ exists and $f^{(i+1)}(0) = 0$. □

A.2 Lemma: *Let $h\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m \in C^0$, $(\lambda_0, y_0) \in h^{-1}(0)$, and for any $\delta \in \mathbb{R}$:*

$$\begin{aligned}\Psi_\delta^+ &= \{(\lambda,y)\in\mathbb{R}\times\mathbb{R}^m | h(\lambda,y) = 0 \wedge \lambda_0 < \lambda < \delta\}\\ \textit{and}\quad \Psi_\delta^- &= \{(\lambda,y)\in\mathbb{R}\times\mathbb{R}^m | h(\lambda,y) = 0 \wedge \delta < \lambda < \lambda_0\}\end{aligned}$$

Then
(λ_0, y_0) is a right (resp. left) limit point of $h^{-1}(0)$ $\Longleftrightarrow$

$$\forall_{\lambda_+>\lambda_0}\,\exists_{N_0\in\mathcal{U}((\lambda_0,y_0))}\,\forall_{N\in\mathcal{U}(p),N\subseteq N_0}\quad N\cap\Psi_{\lambda_+}^+ = \varnothing \tag{12}$$

$$(\textit{resp.}\quad \forall_{\lambda_-<\lambda_0}\,\exists_{N_0\in\mathcal{U}((\lambda_0,y_0))}\,\forall_{N\in\mathcal{U}(p),N\subseteq N_0}\quad N\cap\Psi_{\lambda_-}^- = \varnothing)$$

□

Proof: We show the assertion for right limit points only. $\Psi_{\lambda_+}^+ = h^{-1}(0)\cap(]\lambda_0,\lambda_+[\times\mathbb{R}^m)$
" $\Longleftarrow$ ": Choose $N \subseteq B(\lambda_0,\varepsilon)\times\mathbb{R}^m\cap N_0$, $N\in\mathcal{U}((\lambda_0,y_0))$ with ε sufficient small.[12] $\Longrightarrow$
$(]\lambda_0,\lambda_+[\times\mathbb{R}^m)\cap N = ((\lambda_0+\mathbb{R}_+)\times\mathbb{R}^m)\cap N$
" $\Longrightarrow$ ":

(i) (λ_0, y_0) is a right limit point of $h^{-1}(0) \Longrightarrow$

$$\forall_{\lambda_+>\lambda_0}\,\exists_{N_0\in\mathcal{U}((\lambda_0,y_0))}\, N_0\cap\Psi_{\lambda_+}^+ = \varnothing \tag{13}$$

since $]\lambda_0,\lambda_+[\times\mathbb{R}^m \subseteq (\lambda_0+\mathbb{R}_+)\times\mathbb{R}^m$.

(ii) (13) $\Longrightarrow$ (12) since $N\subseteq N_0 \Longrightarrow N\cap\Psi_{\lambda_+}^+ \subseteq N_0\cap\Psi_{\lambda_+}^+ = \varnothing$ □

[12] $B(m,r)$ denotes the open ball with center m and radius r.

References

[1] Chua, L.O./Deng, A.-C. "Impasse Points. Part I:Numerical Aspects", "Impasse Points. Part II: Analytical Aspects", Int. J. Circ. Theor. Appl., Vol. 17, 213-235 (1989), 271-282 (1989)

[2] Dieudonné, J. "Foundations of Modern Analysis", 7th. ed., Academic Press, New York, London, 1968

[3] Griepentrog, E./März, R. "Differential-Algebraic Equations and Their Numerical Treatment", Leipzig: BSB Teubner 1986, 1.Aufl. (Teubner-Texte zur Mathematik; 88)

[4] Reich, S. "On a Geometrical Interpretation of Differential- Algebraic Equations", Circ. Sys. Sign. Proc. J., Vol. 9, No. 4, 1990, 367-382

19 July 1993
ET-IEE-1-1993, corrected (3/29/94)