On properties of solutions of differential-algebraic equations

Gunther Reißig

24 January 1994

Abstract

Impasse point is a phenomenon of Differential-Algebraic Equations (DAEs), which also occurs in the analysis of electrical networks. It is usually characterized by the condition that solutions of the DAE in question cannot be continued beyond this point. However, it turns out that several classes of impasse points, each of which represents different behavior of the DAE, do exist. In this paper, relations between these classes and between impasse points and pseudo limit points are examined for the first time. Sufficient conditions for a point to be an impasse are given and the existence of maximal continuated solutions is proved. Considerations include non-autonomous and degenerate DAEs as well as non-differentiable solutions. Further, not any set occuring in the analysis of DAEs is assumed to be a manifold.

Als Manuskript gedruckt.

Technische Universität Dresden

Herausgeber: Der Rektor

On properties of solutions of differential-algebraic equations

by Dipl.-Ing. Gunther Reißig

Lehrstuhl für Theoretische Elektrotechnik
Prof. Dr.-Ing. habil. A. Reibiger

Fakultät Elektrotechnik

ET-IEE-2-1994

Abstract

Impasse point is a phenomenon of Differential-Algebraic-Equations (DAEs), which also occurs in the analysis of electrical networks. It is usually characterized by the condition that solutions of the DAE in question cannot be continued beyond this point. However, it turns out that several classes of impasse points, each of which represents different behavior of the DAE, do exist. In this paper, relations between these classes and between impasse points and pseudo limit points are examined for the first time. Sufficient conditions for a point to be an impasse are given and the existence of maximal continuated solutions is proved. Considerations include non-autonomous and degenerate DAEs as well as non-differentiable solutions. Further, not any set occuring in the analysis of DAEs is assumed to be a manifold.

1 Introduction

Impasse point is a phenomenon of Differential-Algebraic-Equations (DAE). It occurs in the analysis of electrical networks and has to be dealt with if the global equations of state of the network in question do not exist.

As an example, consider the simple normalized circuit shown in Fig. 1. Let branch L be an inductor of inductance 1 and D be a tunnel diode with voltage-current-relation (VCR) G_D, which is shown in Fig. 1 as well. The analysis of this network leads to the following system:

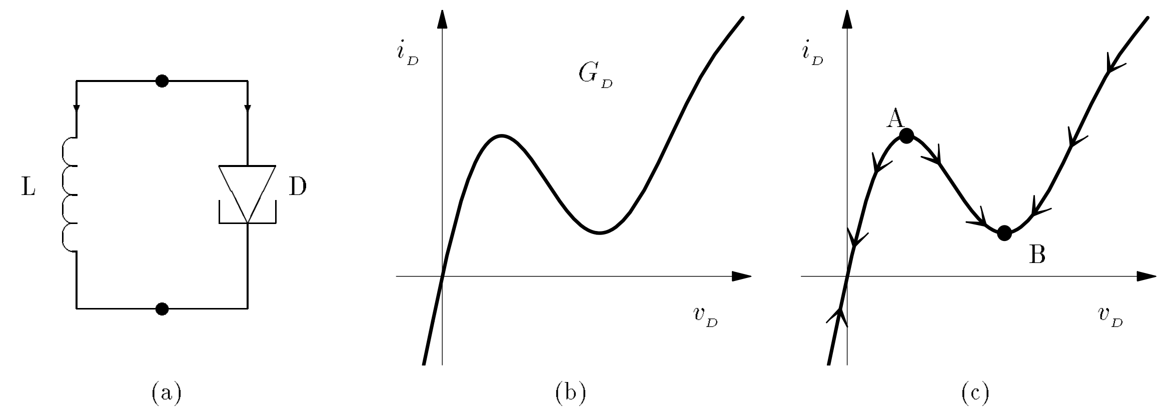


Figure 1: (a) Simple network, the global equations of state of which do not exist. (b) VCR of branch D. (c) Illustration of the solutions of system (1).

$$\begin{aligned} i'_D(t) &= -v_D(t) \\ (v_D(t), i_D(t)) &\in G_D \end{aligned} \tag{1}$$

That means that $\langle v_L, v_D, i_L, i_D\rangle$ is a solution of the network iff $i_L = -i_D$, $v_L = v_D$, and $\langle v_D, i_D\rangle$ is a solution of (1)[1]. The point is that a tunnel diode is not current-controlled, namely, G_D^{-1} is not a function. Therefore, the analysis of (1) cannot be reduced to the analysis of some ordinary differential equation (ODE).

We now observe the following:

(i) For any $p \in G_D \setminus \{A, B\}$ there is a local solution of (1) passing through p.

(ii) There is no solution of (1) passing through A, nor through B.

(iii) There are non-continuable solutions approaching A (resp. B) in backward (resp. forward) time direction, "reaching" A (resp. B) at some finite time.

Because of these phenomena, A (resp. B) is called *backward* (resp. *forward*) *impasse point* [2].

In the following sections we prove the existence of *maximal continuated solutions*, define several classes of *impasse points*, examine relations between these classes, and give sufficient conditions for a point to be an impasse.

In the last section the term *pseudo limit point*, a generalization of *limit point* [2], is introduced and its relation to impasse points is investigated.

Our considerations include non-autonomous and degenerate DAEs as well as non-differentiable solutions. Further, not any set occuring in the analysis of the DAEs we deal with is assumed to be a manifold.

Throughout the paper we will not deal with "Jump Behavior".

2 Basic Definitions

2.1. Definition (DAE, ADAE) *Let $n \in \mathbb{N}$, $m, k \in \mathbb{Z}_+$, $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be open sets and $T \subseteq \mathbb{R}$ open and connected*[2]*. Let f and g be functions. If $f\colon X \times Y \times T \to \mathbb{R}^n \in C^0$ and $g\colon X \times Y \times T \to \mathbb{R}^k \in C^0$, the system*

$$\begin{aligned} \dot{x}(t) &= f(x(t), y(t), t) \\ 0 &= g(x(t), y(t), t) \end{aligned} \tag{2}$$

is called a Differential-Algebraic-Equation (DAE). [3]
If $f\colon X \times Y \to \mathbb{R}^n \in C^0$ and $g\colon X \times Y \to \mathbb{R}^k \in C^0$, the system

$$\begin{aligned} \dot{x} &= f(x, y) \\ 0 &= g(x, y) \end{aligned} \tag{3}$$

is called an autonomous Differential-Algebraic-Equation (ADAE). [3] □

2.2. Definition (Solution) *Consider DAE (2) or ADAE (3) and let $(x_0, y_0) \in X \times Y$, and $t_0 \in \mathbb{R}$ ($t_0 \in T$ in case of DAE (2)). Let further $x\colon I \to X \in C^0$, $y\colon I \to Y \in C^0$.*
$\varphi = \langle x, y\rangle$ is said to be a solution of (2) *(resp. (3))*$:\Longleftrightarrow$

(i) $I \subseteq T$ (resp. $I \subseteq \mathbb{R}$) is open and connected.

(ii) $x \in C^1$

(iii) $\forall_{t \in I}(\dot{x}(t) = f(x(t), y(t), t) \wedge 0 = g(x(t), y(t), t))$
resp. $\forall_{t \in I}(\dot{x}(t) = f(x(t), y(t)) \wedge 0 = g(x(t), y(t)))$

The solution set S of (2) *(resp. (3)) is defined by*

$$S := \{\psi |\ \psi \text{ is a solution of } (2)\ (\text{resp. } (3))\}$$

The C^1-solution set S_{C^1} of (2) *(resp. (3)) is defined by $S_{C^1} := S \cap C^1$. Elements of S_{C^1} are called* C^1-solutions of (2) *(resp. (3)).*

[1]Throughout the paper we rely on the notation of [8], with only a few exceptions: G_D^{-1} is the inverse relation of G_D, i.e., $xG_D^{-1}y :\Longleftrightarrow yG_Dx$. By $\langle x, y\rangle$ we denote the function $M \to X \times Y : t \mapsto (x(t), y(t))$ if $x\colon M \to X$ and $y\colon M \to Y$.

[2]$\mathbb{N}$ is the set of natural numbers, and $\mathbb{Z}_+$ is the set of nonnegative integers. If $i \in \mathbb{Z}_+$ and unless the following abbreviation leads to misunderstandings, C^i and $C^i(E)$ are used instead of $C^i(E, F)$ and denote the set of i times continuously differentiable (resp. continous) mappings from E in F.

[3] In case $m = k = 0$ we identify (2) and (3) with ODEs $\dot{x}(t) = f(x(t), t)$ and $\dot{x} = f(x)$, respectively.

The solution set $S_{(x_0,y_0,t_0)}$ *(resp.* C^1-solution set $S_{C^1,(x_0,y_0,t_0)}$*)* of (2) *and* (3) according to the initial condition (x_0, y_0, t_0) *is defined as follows:*

$$S_{(x_0,y_0,t_0)} := \{\psi \in S |\ \psi(t_0) = (x_0, y_0)\}$$
$$S_{C^1,(x_0,y_0,t_0)} := S_{(x_0,y_0,t_0)} \cap C^1$$

Elements of $S_{(x_0,y_0,t_0)}$ *(resp.* $S_{C^1,(x_0,y_0,t_0)}$*) are called* solutions *(resp.* C^1-solutions*)* of (2) *and* (3) passing through (x_0, y_0) at t_0. $\square$

2.3. Definition (Continuation of solutions) *Consider DAE (2) or ADAE (3) and let S and S_{C^1} be the sets defined in 2.2.. Let further $\alpha \in S$.*
β is called

continuation of α $:\Longleftrightarrow \beta \in S \wedge \alpha \subseteq \beta$ [4]

C^1-continuation of α $:\Longleftrightarrow \beta \in S_{C^1} \wedge \alpha \subseteq \beta$

α is called

non-continuable to the right $:\Longleftrightarrow \forall_{\beta\in S}(\alpha \subseteq \beta \Longrightarrow \sup \mathrm{dom}\, \beta = \sup \mathrm{dom}\, \alpha)$

non- C^1-continuable to the right $:\Longleftrightarrow \forall_{\beta\in S_{C^1}}(\alpha \subseteq \beta \Longrightarrow \sup \mathrm{dom}\, \beta = \sup \mathrm{dom}\, \alpha)$

non-continuable to the left $:\Longleftrightarrow \forall_{\beta\in S}(\alpha \subseteq \beta \Longrightarrow \inf \mathrm{dom}\, \beta = \inf \mathrm{dom}\, \alpha)$

non- C^1-continuable to the left $:\Longleftrightarrow \forall_{\beta\in S_{C^1}}(\alpha \subseteq \beta \Longrightarrow \inf \mathrm{dom}\, \beta = \inf \mathrm{dom}\, \alpha)$

non-continuable $:\Longleftrightarrow \forall_{\beta\in S}(\alpha \subseteq \beta \Longrightarrow \alpha = \beta)$

non- C^1-continuable $:\Longleftrightarrow \forall_{\beta\in S_{C^1}}(\alpha \subseteq \beta \Longrightarrow \alpha = \beta)$ $\square$

Obviously, non-continuability is equivalent to non-continuability to the left and to the right. Similar to the case of ODEs [5], there exist maximal continuated solutions:

2.4. Lemma (Maximal continuated solutions) *Consider DAE (2) or ADAE (3) and let S and S_{C^1} be the sets defined in 2.2.. Suppose that $\alpha \in S$ (resp. $\alpha \in S_{C^1}$). Then*

$$\exists_\beta\ \beta \textit{ is a non-continuable continuation of } \alpha$$

and

$$\exists_\beta\ \beta \textit{ is a non-}C^1\textit{-continuable } C^1\textit{-continuation of } \alpha,$$

respectively. $\square$

Proof: E be the set of continuations of α. Clearly, $(E, \subseteq)$ is an ordered set (see Definition A.1.). Let now $(C, \subseteq)$ be any chain, $C \subseteq E$, and $\tilde{\beta} := \bigcup_{\gamma\in C} \gamma$.[4]

$\bigcup_{\gamma\in C \wedge t\in \mathrm{dom}\,\gamma} \gamma(t)$ contains exacly one element for all $t \in \bigcup_{\gamma\in C} \mathrm{dom}\,\gamma$, because $(C, \subseteq)$ is a chain. Therefore, the definition of $\tilde{\beta}$ makes sense. Since $\tilde{\beta}$ is a continuation of α and of any $\gamma \in C$, it is an upper bound of C in E. By Zorn's Lemma (A.2.), there is a maximal element β of $(E, \subseteq)$, which is obviously a non-continuable continuation of α.
The case of C^1-continuations is done in exactly the same way. $\square$

[4] Functions are considered to be sets of pairs of form $(x, f(x))$.

If we had defined solutions to be functiones over *closed* intervalls [4], a non-continuable solution would not exist in general, not even in case of ODEs.

2.5. Definition (State set, Consistent Initial Value) *Consider DAE (2) (resp. ADAE (3)). Using the sets defined in 2.2., the* state set P *and the* C^1-state set P_{C^1} of (2) *(resp.* (3)*) are defined by*

$$P := \{z \in X \times Y \times T |\ S_z \neq \emptyset\} \qquad P_{C^1} := \{z \in X \times Y \times T |\ S_{C^1,z} \neq \emptyset\}.$$

resp.

$$P := \{(x, y) \in X \times Y |\ S_{(x,y,0)} \neq \emptyset\} \qquad P_{C^1} := \{(x, y) \in X \times Y |\ S_{C^1,(x,y,0)} \neq \emptyset\}.$$

Elements of P *and* P_{C^1} *are called* consistent *and* C^1-consistent initial values of (2) *(resp.* (3)). □

3 Impasse Points

3.1. Definition (Impasse point) *Consider DAE (2) (resp. ADAE (3)) and let* $Q := X \times Y \times T$, $p := (x_0, y_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ *(resp.* $Q := X \times Y$, $p = (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, *and* $t_0 = 0$*). Using the sets defined in section 2, p is called*

impasse point *(resp.* C^1-impasse point*)* of the 1st kind *(IP-1, resp.* C^1*-IP-1)*:$\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \qquad \textit{resp.} \qquad p \in g^{-1}(0) \setminus P_{C^1}$$

impasse point *(resp.* C^1-impasse point*)* of the 2nd kind *(IP-2, resp.* C^1*-IP-2)*:$\Longleftrightarrow$

$$p \in (\overline{P} \cap Q) \setminus P \qquad \textit{resp.} \qquad p \in (\overline{P_{C^1}} \cap Q) \setminus P_{C^1})^5$$

forward *(resp.* C^1-forward*)* impasse point of the 1st kind *(FIP-1, resp.* C^1*-FIP-1)*:$\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \wedge \exists_{\psi \in S}(\sup \operatorname{dom} \psi = t_0 \ \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

resp.

$$p \in g^{-1}(0) \setminus P_{C^1} \wedge \exists_{\psi \in S_{C^1}}(\sup \operatorname{dom} \psi = t_0 \ \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

forward *(resp.* C^1-forward*)* impasse point of the 2nd kind *(FIP-2, resp.* C^1*-FIP-2)*:$\Longleftrightarrow$

$$p \in P \wedge \exists_{\psi \in S}(\psi \textit{ is non-continuable to the right} \wedge \sup \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

resp.

$$p \in P_{C^1} \wedge \exists_{\psi \in S_{C^1}}(\psi \textit{ is non-}C^1\textit{-continuable to the right} \wedge \sup \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

backward *(resp.* C^1-backward*)* impasse point of the 1st kind *(BIP-1, resp.* C^1*-BIP-1)*:$\Longleftrightarrow$

$$p \in g^{-1}(0) \setminus P \wedge \exists_{\psi \in S}(\inf \operatorname{dom} \psi = t_0 \ \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

resp.

$$p \in g^{-1}(0) \setminus P_{C^1} \wedge \exists_{\psi \in S_{C^1}}(\inf \operatorname{dom} \psi = t_0 \ \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

backward *(resp.* C^1-backward*)* impasse point of the 2nd kind *(BIP-2, resp.* C^1*-BIP-2)*:$\Longleftrightarrow$

$$p \in P \wedge \exists_{\psi \in S}(\psi \textit{ is non-continuable to the left} \wedge \inf \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

resp.

$$p \in P_{C^1} \wedge \exists_{\psi \in S_{C^1}}(\psi \textit{ is non-}C^1\textit{-continuable to the left} \wedge \inf \operatorname{dom} \psi = t_0 \wedge \lim_{t \to t_0} \psi(t) = (x_0, y_0))$$

$\square$

In the sequel, the sets of impasse points of the first kind, of the second kind, of forward impasse points of the 1st kind, of the 2nd kind, of backward impasse points of the 1st kind, and of the 2nd kind (resp. the corresponding kinds of C^1-impasse points) will be refered to by I_1, I_2, $I_{F,1}$, $I_{F,2}$, $I_{B,1}$, and $I_{B,2}$ (resp. $I_{C^1,1}$, $I_{C^1,2}$, $I_{C^1,F,1}$, $I_{C^1,F,2}$, $I_{C^1,B,1}$, and $I_{C^1,B,2}$).

3.2. Lemma: *Consider DAE (2) or ADAE (3). Then*

(i) $I_{B,1} \cup I_{F,1} \subseteq I_2 \subseteq I_1$

(ii) $I_{C^1,B,1} \cup I_{C^1,F,1} \subseteq I_{C^1,2} \subseteq I_{C^1,1}$

(iii) $I_1 \subseteq I_{C^1,1}$

(iv) $I_{F,2} = I_{B,2} = \emptyset$

(v) $(I_{C^1,F,2} \cup I_{C^1,B,2}) \cap I_{C^1,1} = \emptyset$ $\square$

Proof: We prove the assertions only for DAE (2):

(i) Let $p = (x_0, y_0, t_0) \in I_{B,1}$, then $p \in g^{-1}(0) \setminus P$ and $\exists_{\psi=\langle x,y\rangle \in S} \sup \operatorname{dom} \psi = t_0 \wedge \lim_{t\to t_0} \psi(t) = (x_0, y_0)$, i.e., $\lim_{t\to t_0}(x(t), y(t), t) = p$. Since $\operatorname{im}\langle x, y, \operatorname{id}|_{\operatorname{dom}\psi}\rangle \subseteq P$, we get $p \in \overline{P}$, i.e., $p \in g^{-1}(0)$, $p \notin P$, and $p \in \overline{P} \implies p \in (\overline{P} \cap g^{-1}(0)) \setminus P \subseteq (\overline{P} \cap X \times Y \times T) \setminus P = I_2$.
Let $p \in I_2 = (\overline{P} \cap X \times Y \times T) \setminus P$. Since $g^{-1}(0)$ is closed (in $X \times Y \times T$) and $P \subseteq g^{-1}(0)$, we get $\overline{P} \cap X \times Y \times T \subseteq g^{-1}(0) \implies p \in g^{-1}(0) \setminus P = I_1$.

(ii) Equivalent to (i).

(iii),(v) trivial.

(iv) Let, without loss of generality, $p = (x_0, y_0, t_0) \in I_{F,2}$, $\varphi\colon]t_0 - \varepsilon, t_0 + \varepsilon[\to X \times Y \in S_p$, $\psi\colon]t_0 - \varepsilon, t_0[\to X \times Y \in S$ for some $\varepsilon > 0$, $\lim_{t\to t_0} \psi(t) = (x_0, y_0)$, and ψ be non-continuable to the right.
Set $\beta := \langle x, y\rangle := \psi \cup \varphi|_{[t_0,t_0+\varepsilon[}$, i.e.,

$$\beta\colon]t_0 - \varepsilon, t_0 + \varepsilon[\to X \times Y : t \mapsto \left\{ \begin{array}{ll} \psi(t) & \text{for } t < t_0 \\ \varphi(t) & \text{otherwise} \end{array} \right\}.$$

Obviously, $\beta \in C^0$ and $0 = g(x(t), y(t), t)$ for all $t \in \operatorname{dom}\beta$.
We now have to show that $x \in C^1$ and that $\dot{x}(t) = f(x(t), y(t), t)$ for all $t \in \operatorname{dom}\beta$. The idea is to deal with an integral equation

$$\xi(t) = \xi(t_0) + \int_{t_0}^{t} F(\xi(\tau), \tau) d\tau \tag{4}$$

instead of an ODE

$$\dot{\xi}(t) = F(\xi(t), t) \tag{5}$$

where $F\colon X \times T \to \mathbb{R}^n \in C^0$. As is well known, (5) is equivalent to (4) in the following sense [1, (6.5a)]:

[5] Here, $\overline{P}$ denotes the closure of P in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ (resp. $\mathbb{R}^n \times \mathbb{R}^m$) endowed with the topology induced by some norm. If P^a is the closure of P in the space Q with the topology induced by $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ (resp. $\mathbb{R}^n \times \mathbb{R}^m$), one gets $\overline{P} \cap Q = P^a$ etc.

Let $\xi: I \to X \in C^0$, $I \subseteq T$ an open intervall, and $t_0 \in I$. Then ξ is a C^0-solution of (4) iff ξ is a C^1-solution of (5).
In the concrete, since $\psi = \beta|_{]t_0-\varepsilon,t_0[}$ is a solution of the DAE in question, we have for all $t \in]t_0-\varepsilon, t_0[$

$$x(t) = x(t_0 - \frac{\varepsilon}{2}) + \int\limits_{t_0-\frac{\varepsilon}{2}}^{t} f(x(\tau), y(\tau), \tau)d\tau \tag{6}$$

Since x, y, and f are continous, both sides of (6) are continous with respect to t and hence, (6) also holds for $t = t_0$. By substituting $x(t_0 - \frac{\varepsilon}{2})$ we get

$$x(t) = x(t_0) + \int\limits_{t_0}^{t} f(x(\tau), y(\tau), \tau)d\tau \tag{7}$$

for all $t \in]t_0 - \varepsilon, t_0]$.
Let $\varphi = \langle \tilde{x}, \tilde{y} \rangle$. Then, for all $t \in]t_0 - \varepsilon, t_0 + \varepsilon[$, $\tilde{x}$ fulfills an integral equation analogous to (7), which is, for $t \in [t_0 - \varepsilon, t_0[$, exactly the same as (7). Hence, (7) holds for all $t \in \operatorname{dom}\beta$ and thus, $\dot{x}(t) = f(x(t), y(t), t)$ also holds. So, β is a solution of the DAE in question in contradiction to the assumption that ψ is non-continuable to the right. $\square$

As we will see in this section, the kinds of impasse points defined above are in general different from each other. We will show, at least, that $I_1 \neq I_{C^1,1}$ (Example 3.9.), $I_{C^1,1} \neq I_{C^1,2}$ and $I_1 \neq I_2$ (Example 3.7.). $I_2 \neq I_{B,1} \cup I_{F,1}$ and $I_{C^1,2} \neq I_{C^1,B,1} \cup I_{C^1,F,1}$ is shown in [7, Example 2.11]. Further, we see that C^1-forward and C^1-backward impasse points of the second kind do exist:

3.3. Example:

$$\begin{aligned} \dot{x} &= 1 \\ 0 &= y\left((x-y)^2 + f(y)\right) \end{aligned}$$

with $f \in C^\infty$, $f|_{\mathbb{R}_-} = 0$ and $f|_{\mathbb{R}_+\setminus\{0\}} > 0$. Then, $\varphi: \mathbb{R} \to \mathbb{R}^2: t \mapsto (t, 0)$ is clearly in S_{C^1}, and we have $(0,0) \in P_{C^1}$. Further, $\psi: \mathbb{R}_- \setminus \{0\} \to \mathbb{R}^2: t \mapsto (t,t)$ is in S_{C^1} as well and $\lim\limits_{\substack{t\to 0\\ t<0}} \psi(t) = (0,0)$. Let $\langle x, y\rangle$ be a proper C^1-continuation of ψ to the right, i.e., $\sup\operatorname{dom} x > 0$. Then $x = \operatorname{id}|_{\operatorname{dom} x}$ and $\forall_{t\geq 0} y(t) = 0$. Thus,

$$\lim_{\substack{t\to 0\\ t>0}} \frac{y(t)-y(0)}{t} = 0 \quad \text{and} \quad \lim_{\substack{t\to 0\\ t<0}} \frac{y(t)-y(0)}{t} = 1.$$

Hence, ψ is non-C^1-continuable to the right and $(0,0)$ is a C^1-FIP-2. $\square$

In case $m = k = 0$ we identify DAE (2) and ADAE (3) with ODEs $\dot{x}(t) = f(x(t), t)$ and $\dot{x} = f(x)$, respectively. As is well known from the theory of ODEs, we get $P = P_{C^1} = X \times T$ (resp. $P = P_{C^1} = X$) in that case and thus, impasse points cannot occur. We consider that to be an important difference between DAEs and ODEs. In the following lemma, situations are examined where the DAE or ADAE in question is locally equivalent to an ODE in some sense:

3.4. Lemma: *Consider DAE (2) (resp. ADAE (3)) and let* $(x_0, y_0, t_0) \in X \times Y \times T$ *(resp.* $(x_0, y_0) \in X \times Y$*). Then*

(i) $U \in \mathcal{U}((x_0, t_0)) \wedge \exists_{h:\, U\to Y \in C^0}(h(x_0,t_0) = y_0 \wedge \forall_{(x,t)\in U} g(x, h(x,t), t) = 0)$
$\implies \{(x, h(x,t), t) |\ (x,t) \in U\} \subseteq P$ [6]

[6] $\mathcal{U}(p)$ denotes the set of open neighbourhoods of point p with respect to the considered space.

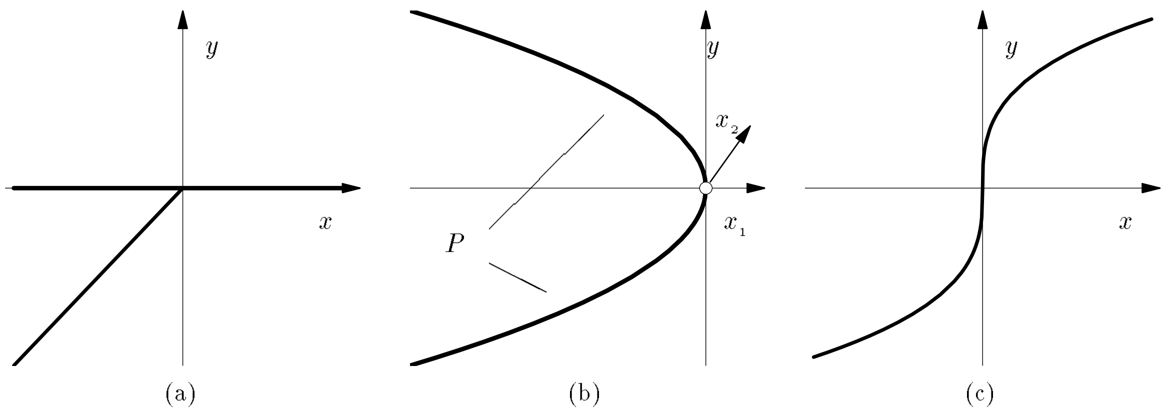


Figure 2: (a) $g^{-1}(0)$ as in Example 3.3. (b) P as in Example 3.7., $g^{-1}(0) = (\mathbb{R} \times \{0\}) \times \mathbb{R}$. (c) $g^{-1}(0)$ as in Example 3.9.

(ii) $U \in \mathcal{U}((x_0, t_0)) \wedge \exists_{h:\, U \to Y \in C^1}(h(x_0, t_0) = y_0 \wedge \forall_{(x,t) \in U} g(x, h(x,t), t) = 0)$
$\Longrightarrow \{(x, h(x,t), t)|\ (x,t) \in U\} \subseteq P_{C^1}$

resp.

(i) $U \in \mathcal{U}(x_0) \wedge \exists_{h:\, U \to Y \in C^0}(h(x_0) = y_0 \wedge \forall_{x \in U} g(x, h(x)) = 0) \Longrightarrow \{(x, h(x))|\ x \in U\} \subseteq P$

(ii) $U \in \mathcal{U}(x_0) \wedge \exists_{h:\, U \to Y \in C^1}(h(x_0) = y_0 \wedge \forall_{x \in U} g(x, h(x)) = 0) \Longrightarrow \{(x, h(x))|\ x \in U\} \subseteq P_{C^1}$

$\Box$

Proof: Only the assertions concerning DAE (2) will be proved:

(i) Let $(x_1, t_1) \in U$. Then $(x_1, t_1) \in Q \times J \subseteq U$, where J is an open intervall and $Q \subseteq X$ is open. Consider the ODE $\dot{x}(t) = \tilde{f}(x(t), t)$, where $\tilde{f}\colon Q \times J \to \mathbb{R}^n\colon (c,t) \mapsto f(c, h(c,t), t) \in C^0$. Obviously, there is a solution $\varphi\colon I \to Q \in C^1$ with $\varphi(t_1) = x_1$ ($I \subseteq J$ is an open interval). It is easy to show that $\psi\colon I \to X \times Y : t \mapsto (\varphi(t), h(\varphi(t), t)) \in S$ [7] and hence, $(\varphi(t_1), h(\varphi(t_1), t_1), t_1) = (x_1, h(x_1, t_1), t_1) \in P$.

(ii) Assume that $h \in C^1$. In addition to (i) we have $h(\varphi(\cdot), \cdot) \in C^1$ and hence, $\psi \in S_{C^1}$. $\Box$

Since h does not need to be unique, Lemma 3.4. is slightly more general than the statements in [2, Lemma 1 and 2], as can be easily seen from Example 3.3..

3.5. Remark: There are simple sufficient conditions for the existence of a function h as required in Lemma 3.4.:
Consider DAE (2) (resp. ADAE (3)) and let $p = (x_0, y_0, t_0) \in X \times Y \times T$ (resp. $p = (x_0, y_0) \in X \times Y$). Then

(i) $V \in \mathcal{U}(p) \wedge D_2 g$ exists as a partial F-derivative on $V \wedge D_2 g$ is continous at $p \wedge D_2 g(p)$ is surjective $\Longrightarrow$ There is a $U \in \mathcal{U}((x_0, t_0))$ (resp. $U \in \mathcal{U}(x_0)$) that meets *(i)* in Lemma 3.4..

(ii) $V \in \mathcal{U}(p) \wedge g|_V \in C^1 \wedge D_2 g(p)$ *is surjective* $\Longrightarrow$ There is a $U \in \mathcal{U}((x_0, t_0))$ (resp. $U \in \mathcal{U}(x_0)$) that meets *(ii)* in Lemma 3.4.. $\Box$

Proof: The aim is to prove the existence of certain implicit functions. Concerning (ii), [8, Theorem 4.H] immediately applies, so we show (i) for DAE (2):
Let $Y_1 := \ker D_2 g(p)$ and $G\colon X \times T \times Y_1 \times Y_1^\perp \to \mathbb{R}^k\colon (x,t,y_1,y_2) \mapsto g(x, y_0 + y_1 + y_2, t)$. Obviously, $D_4 G$ exists as a partial F-derivative, namely, $D_4 G(x,t,y_1,y_2) = D_2 g(x, y_0 + y_1 + y_2, t)|_{Y_1^\perp}$ for all (x,t,y_1,y_2) in a neighbourhood of $(x_0,t_0,0,0)$ corresponding to V. Further, $D_4 G$ is continous at $(x_0,t_0,0,0)$ and $D_4 G(x_0,t_0,0,0)$ is bijective. Applying the Implicit Function Theorem B.1., we get[7]

$$\exists_{\tilde U \in \mathcal{U}((x_0,t_0,0))} \exists!_{\tilde h\colon \tilde U \to Y_1^\perp \in C^0} (\tilde h(x_0,t_0,0) = 0 \ \wedge \forall_{(x,t,y_1)\in \tilde U} G(x,t,y_1,\tilde h(x,t,y_1)) = 0),$$

i.e., $\forall_{(x,t,y_1)\in\tilde U} g(x, y_0 + y_1 + \tilde h(x,t,y_1), t) = 0$. Set $h(x,t) := y_0 + \tilde h(x,t,0)$ for (x,t) in some neighbourhood U of (x_0,t_0). □

Using ideas from Lemma 3.4. and Remark 3.5., one could easily find sufficient conditions for local uniqueness of solutions of DAEs and ADAEs. Instead of doing so, we will focus on criteria for a point to be an impasse:

Let us consider ADAE (3), let $p \in g^{-1}(0)$, and assume that $Dg(p)$ has full rank. Then $g^{-1}(0)$ is locally a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ and $\dot\varphi(t_0)$ has to lie in the tangent space of $g^{-1}(0)$ at p for any C^1-solution $\varphi = \langle x, y\rangle$ passing through p at t_0 [6]. So we have

$$\dot\varphi(t_0) = (f(p), \dot y(t_0)) \in \ker Dg(p)$$

and hence

$$D_1 g(p) f(p) \in \operatorname{im} D_2 g(p). \tag{8}$$

Conversely, if (8) does not hold, p is clearly a C^1-IP-1. As we will see in the sequel, this conclusion is valid whether or not $g^{-1}(0)$ is a manifold.

3.6. Definition: *Consider DAE (2) (resp. ADAE (3)), let $p \in g^{-1}(0)$, and assume that g is C^1 on some neighbourhood of p. Then the* tangential property *is defined as follows:*

Tg(p) $\quad D_1 g(p) f(p) + D_3 g(p) 1 \in \operatorname{im} D_2 g(p)$

resp.

Tg(p) $\quad D_1 g(p) f(p) \in \operatorname{im} D_2 g(p)$ □

3.7. Example:

$$\begin{aligned} \dot x_1 &= 1 \\ \dot x_2 &= x_1 + y^2 \\ 0 &= x_2 \end{aligned}$$

Obviously, for all $(x_{1,0}, y_0) \in \mathbb{R}\times\mathbb{R}$ satisfying $x_{1,0} < 0$ and $x_{1,0} + y_0^2 = 0$, $\varphi\colon]x_{1,0}, -x_{1,0}[\to \mathbb{R}^2 \times \mathbb{R}$ with

$$\varphi(t) := \left((x_{1,0} + t, 0), \sqrt{-x_{1,0} - t}\begin{cases} 1 & \text{if } y_0 > 0 \\ -1 & \text{otherwise}\end{cases}\right)$$

is in S_{C^1}, and hence, $\{((x_1,0),y) |\ x_1 < 0 \wedge x_1 + y^2 = 0\} \subseteq P_{C^1}$.
Let now $\langle\langle x_1, x_2\rangle, y\rangle$ be in S. Then $x_2 = 0$ and, by that fact, $0 = \dot x_2 = x_1 + y^2$, and hence, $P_{C^1} \subseteq \{((x_1,0),y) |\ x_1 + y^2 = 0\}$.
Assume now $\langle\langle x_1,x_2\rangle, y\rangle \in S_{((0,0),0,0)}$. Then, there is some $\varepsilon > 0$ that $]-\varepsilon,\varepsilon[\subseteq \operatorname{im} x_1$, because $\dot x_1(0) = 1$. Contradiction.
So, we get:

[7] ∃! means "There is one and only one … ".

(i) $P = P_{C^1} = \{((x_1,0),y) \in \mathbb{R}^2 \times \mathbb{R} |\ x_1 + y^2 = 0 \wedge x_1 < 0\}$[8]

(ii) $I_{F,1} = I_{C^1,F,1} = I_2 = I_{C^1,2} = \{((0,0),0)\}$

(iii) $I_{B,1} = I_{C^1,B,1} = I_{B,2} = I_{C^1,B,2} = I_{F,2} = I_{C^1,F,2} = \emptyset$

(iv) $I_1 = I_{C^1,1} = g^{-1}(0) \setminus P = \{((x_1,0),y) \in \mathbb{R}^2 \times \mathbb{R} |\ x_1 \geq 0 \vee x_1 + y^2 \neq 0\}$

Note that, although $((0,0),0)$ is in $I_{F,1}$ as well as in $I_{C^1,F,1}$, it has the tangential property $Tg((0,0),0)$:

$$D_1 g((0,0),0) f((0,0),0) = 0 \in \{0\} = \operatorname{im} D_2 g((0,0),0)$$

Therefore, this example contradicts [2, Lemma 3], which conjectured that[9]

$$p \textit{ is an impasse point} \Longrightarrow \neg Tg(p).$$

□

3.8. Lemma ([7]) *Consider DAE (2) (resp. ADAE (3)), let $p \in g^{-1}(0)$, and assume that g is C^1 on some neighbourhood of p. Then*

$$\neg Tg(p) \Longrightarrow p \in I_{C^1,1}.$$

□

The proof [7] is a simple application of the chain rule. Further, Lemma 3.8. is a special case of Lemma 3.10..

Note that, under the assumptions of Lemma 3.8., p does neither need to be in I_1 (Example 3.9.), nor in $I_{C^1,2}$ (Example 3.7.).

3.9. Example:

$$\begin{aligned} \dot{x} &= 1 \\ 0 &= y^3 - x \end{aligned}$$

Set $h\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \sqrt[3]{|x|}\operatorname{sign}(x)$ and $\tilde{h} := h|_{\mathbb{R}\setminus\{0\}}$. By Lemma 3.4.(ii) we get $\{(x,\tilde{h}(x))|\ x \in \mathbb{R}\setminus\{0\}\} = g^{-1}(0)\setminus\{(0,0)\} \subseteq P_{C^1}$, since $\tilde{h} \in C^1$. By Lemma 3.4.(i) we have $(0,0) \in P$, since $h \in C^0$. By Lemma 3.8. we get that $(0,0) \notin P_{C^1}$, because $Dg(0,0)(x,y) = -x$ and $D_1 g(0,0) f(0,0) = -1 \notin \{0\} = \operatorname{im} D_2 g(0,0)$. So we have $P_{C^1} = g^{-1}(0)\setminus\{(0,0)\}$, $P = g^{-1}(0)$, $I_{C^1,1} = \{(0,0)\}$, and $I_1 = \emptyset$. □

In the situation of Lemma 3.8., no solution passing through (x_0, y_0) at t_0 is continously differentiable. Concerning Example 3.9., it is easy to show that a solution passing through $(0,0)$ at 0 could not even be Lipschitz continous on some neighbourhood of 0. As the following statement shows, this is not an accidental observation:

3.10. Lemma: *Consider DAE (2) (resp. ADAE (3)), let $\|\cdot\|$ be some norm on $\mathbb{R}^m$, $p = (x_0, y_0, t_0) \in g^{-1}(0)$ (resp. $p = (x_0, y_0) \in g^{-1}(0)$, $t_0 \in \mathbb{R}$), and assume that g is C^1 on some neighbourhood of p. If p does not have the tangential property $Tg(p)$ and $\langle x, y\rangle \in S_{(x_0,y_0,t_0)}$, then*

$$\lim_{t \to t_0} \frac{\|y(t) - y_0\|}{|t - t_0|} = \infty.$$

□

[8] We could use here the procedure given in [6] to get P_{C^1}, but want to get P as well.

[9] $g(p) = 0 \wedge \neg Tg(p)$ can be shown to be equivalent to $p \in S_3$, with S_3 as in [2].

Proof: The statement is proved for DAE (2) only:
Assume there is some $L \in \mathbb{R}$ and some sequence $(t_n)_{n\in\mathbb{N}}$, that $t_n \in \operatorname{dom} y \setminus \{t_0\}$ for all n, and $t_n \to t_0$, and

$$\forall_{n\in\mathbb{N}} \quad \frac{\|\Delta y_n\|}{|\Delta t_n|} \leq L \tag{9}$$

contrary to the assertion, where $\Delta y = y(t) - y_0$, $\Delta y_n = y(t_n) - t_0$, $\Delta t = t - t_0$, and $\Delta t_n = t_n - t_0$. Choose $\lambda > 0$ that $\eta\colon B(t_0, \lambda) \times Y \to \mathbb{R}^k\colon (t,z) \mapsto g(x(t), z, t)$ is in C^1. Obviously, $\eta(t, y(t)) = 0$ as long as $t \in \operatorname{dom} y \cap B(t_0, \lambda)$, and

$$D\eta(t_0, y_0)(t, z) = Dg(p)(t \cdot f(p), z, t).$$

In view of the definition of the F-derivative, one yields[10]

$$\forall_{\varepsilon>0}\exists_{r>0}\forall_{(t,z)\in B((t_0,y_0),r)} \ \|\eta(t,z) - Dg(p)(\Delta t \cdot f(p), z - y_0, \Delta t)\| \leq \varepsilon \|(\Delta t, z - y_0)\|.$$

Especially for $z := y(t_n)$ and $t := t_n$ we get

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n>N} \ \|Dg(p)(\Delta t_n \cdot f(p), \Delta y_n, \Delta t_n)\| \leq \varepsilon \|(\Delta t_n, \Delta y_n)\| \leq \varepsilon M |\Delta t_n|$$

for some $M > 0$, the right inequality of which is yielded by (9) and equivalence of norms [3, 3.20.]. Thus,

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n>N} \ \underbrace{\|v + D_2 g(p)\frac{\Delta y_n}{\Delta t_n}\|}_{\delta_n :=} \leq \varepsilon M \tag{10}$$

where $v := D_1 g(p) f(p) + D_3 g(p) 1$, and further[11]

$$\begin{aligned} \delta_n &= \operatorname{dist}(v, -D_2 g(p)\frac{\Delta y_n}{\Delta t_n}) \\ &\geq \inf_{\xi\in\mathbb{R}^m} \operatorname{dist}(v, D_2 g(p)\xi) \\ &\geq \underbrace{\operatorname{dist}(v, \operatorname{im} D_2 g(p))}_{\Delta :=} \end{aligned} \tag{11}$$

Δ is positive since $\operatorname{im}(D_2 g(p))$ is a closed subset of $\mathbb{R}^k$ not containing v. By (10) and (11), $0 < \Delta \leq \varepsilon M$ holds for all $\varepsilon > 0$. Contradiction. □

4 Pseudo Limit Points

In this section we define the term *pseudo limit point*, a generalization of *limit point* ([2], Appendix D), and examine the relation between pseudo limit points and impasse points.

4.1. Definition (Pseudo limit point) *Let $M \subseteq \mathbb{R} \times \mathbb{R}^m$ and $(\lambda_0, y_0) \in M$. (λ_0, y_0) is called* right *(resp.* left*)* pseudo limit point *(PLP) of M* $:\Longleftrightarrow$

$$\forall_{c\in C([0,1],M)} \left(c(0) = (\lambda_0, y_0) \Longrightarrow \exists_{t>0} \pi_\lambda(c(t)) \leq \lambda_0\right)$$

resp.

$$\forall_{c\in C([0,1],M)} \left(c(0) = (\lambda_0, y_0) \Longrightarrow \exists_{t>0} \pi_\lambda(c(t)) \geq \lambda_0\right)$$

where $\pi_\lambda\colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}\colon (\lambda, y) \mapsto \lambda$. □

[10] For simplicity, we denote any used norm by $\|\cdot\|$, which cannot lead to misunderstandings here.

[11] dist denotes the metric according to the norm in $\mathbb{R}^k$. $\operatorname{dist}(A, B)$ is the distance between the sets A and B [8], $\operatorname{dist}(x, B) := \operatorname{dist}(\{x\}, B)$.

4.2. Definition (Cut mapping) *Consider DAE (2) (resp. ADAE (3)) and let* $p = (x_0, y_0, t_0) \in g^{-1}(0)$ *(resp.* $p = (x_0, y_0) \in g^{-1}(0)$*). Let further* $I \subseteq \mathbb{R}$ *be open and connected, so that* $0 \in I$ *and* $x_0 + I \cdot f(p) \subseteq X$*. In case of DAE (2), let in addition* $t_0 + I \subseteq T$*. The mapping*

$$h\colon I \times Y \to \mathbb{R}^k\colon (\lambda, y) \mapsto g(x_0 + \lambda \cdot f(p), y, t_0 + \lambda)$$

resp.

$$h\colon I \times Y \to \mathbb{R}^k\colon (\lambda, y) \mapsto g(x_0 + \lambda \cdot f(p), y)$$

is called cut mapping of (2) *(resp.* (3)*) at* p*.* □

4.3. Remark:

(i) The property of a point to be a PLP of a cut mapping is independent of the choice of I in Definition 4.2..

(ii) Let h be a cut mapping of ADAE (3). Then, any limit point ([2], Appendix D) of $h^{-1}(0)$ is clearly a PLP. The converse is not true, not even if $h^{-1}(0)$ is a C^∞-curve [7, Example 3.3]. □

We now give a criterion about the relation between impasse points and pseudo limit points:

4.4. Theorem: *Consider DAE (2) (resp. ADAE (3)) and let* $p = (x_0, y_0, t_0) \in g^{-1}(0)$ *(resp.* $p = (x_0, y_0) \in g^{-1}(0)$*). Let further* h *be a cut mapping of (2) (resp. (3)) at* p *and the following conditions be satisfied:*

(i) g *is* C^1 *on some neighbourhood of* p

(ii) p *does not have the tangential property* $Tg(p)$

(iii) $\operatorname{rank} D_2 g(p) = k - 1$

Then

$$(0, y_0) \text{ is a left or a right PLP of } h^{-1}(0) \Longrightarrow p \in I_1.$$

□

The trick of the proof of the foregoing theorem is to assume the existence of some $\langle x, y\rangle \in S_{(x_0,y_0,t_0)}$ and then to apply an Implicit Function Theorem to the mapping

$$G\colon (s, t, z) \mapsto g(x_0 + s \cdot f(p), y(t + t_0) + z, s + t_0).$$

Note that, in general, G is not differentiable on some open neighbourhood of $(0, 0, 0)$. So, the longer part of the proof of Theorem 4.4. is to show the F-differentiability of G at $(0, 0, 0)$. Therefore, we put this part into a separate lemma:

4.5. Lemma: *Consider DAE (2) and assume that* g *is* C^1 *on some open neighbourhood* U *of* $p = (x_0, y_0, t_0) \in g^{-1}(0)$*. Let* $\langle x, y\rangle \in S_p$*,* $I := \operatorname{dom}\langle x, y\rangle$*,* $E := \mathbb{R} \times \mathbb{R} \times (\ker D_2 g(p))^\perp$*,* $W \subseteq E$ *an open ball with center* $(0, 0, 0)$*, so that* $t + t_0 \in I$ *and* $(x_0 + s \cdot f(p), y(t + t_0) + z, s + t_0) \in U$ *for all* $(s, t, z) \in W$*. Set*

$$G\colon W \to \mathbb{R}^k\colon (s, t, z) \mapsto g(x_0 + s \cdot f(p), y(t + t_0) + z, s + t_0).$$

Then

(i) $G \in C^0$ *and* $G(0, 0, 0) = 0$*.*

(ii) *G is F-differentiable at* $(0,0,0)$.

(iii) $D_2G(0,0,0)t = -D_1g(p)f(p)t - D_3g(p)t$.

(iv) $D_{1,3}G$ *exists as a partial F-derivative on* W*, is continous, and* $D_{1,3}G(0,0,0)(s,z) = D_1g(p)f(p)s + D_3g(p)s + D_2g(p)z$. □

The proof can be found in Appendix C.

Proof (of Theorem 4.4.) The proof is done for DAE (2) only:
Assume $\langle x,y\rangle \in S_p$ contrary to the assertion and let I, E, W, and G as in Lemma 4.5.. Since $D_{1,3}G(0,0,0)$ is bijective [12](has rank k because of (ii) and (iii)), the Implicit Function Theorem B.1. is applicable and one gets

$$\exists_{r,r_0>0}\forall_{t\in B(t_0,r_0)}\exists!_{(s,z)\in B(0,r)}G(s,t,z) = 0.$$

In the following we denote the point (s,z) corresponding to t by (s(t),z(t)), i.e.,

$$g(x_0 + s(t)f(p), y(t+t_0) + z(t), s(t) + t_0) = 0.$$

Further, $\langle s,z\rangle$ is continous on some neighbourhood of 0, F-differentiable at 0, and

$$D\langle s,z\rangle(0) = -D_{1,3}G(0,0,0)^{-1} \circ D_2G(0,0,0) \;=\; -\Big(v \;\vdots\; D_2g(p)\Big)^{-1} \cdot (-v)$$

$$= \begin{pmatrix} 1 \\ \cdots \\ 0 \end{pmatrix},$$

where $v := D_1g(p)f(p) + D_3g(p)1$. So, it follows that $\dot{s}(0) = 1$ and hence

$$\exists_{t_1>0}\forall_{t\in[0,t_1]}\Big(s(t) > 0 \wedge G(s(t),t,z(t)) = \; g(x_0 + s(t)f(p), y(t+t_0) + z(t), s(t) + t_0) = 0\Big),$$

i.e., $(s(t), y(t+t_0) + z(t)) \in h^{-1}(0)$ for all $t \in [0,t_1]$, contrary to the hypothesis that $(0,y_0)$ is a right PLP of $h^{-1}(0)$. Nor can it be a left PLP by similar arguments. Contradiction. □

4.6. Remark: (i) If we substituted (iii) in the foregoing Theorem by rank $D_2g(p) = k$, the assertion would trivially hold: $D_2g(p)$ is bijective in that case and by the Implicit Function Theorem, $(0,y_0)$ could not be a PLP of $h^{-1}(0)$.

(ii) Condition (ii) in Theorem 4.4. cannot be omitted (Example 4.7.).

(iii) The "$\Longleftarrow$"-part of Theorem 4.4. would not be true, as Example 4.8. shows.

(iv) Although condition (iii) of Theorem 4.4. is used in its proof, examples which fulfill all conditions except (iii) and for which the assertion of Theorem 4.4. is not valid, are not known.

□

[12]Obviously, $D_{2,3}$ is also bijective at $(0,0,0)$, but does not exist on some open neighbourhood of $(0,0,0)$. So we have to go indirectly.

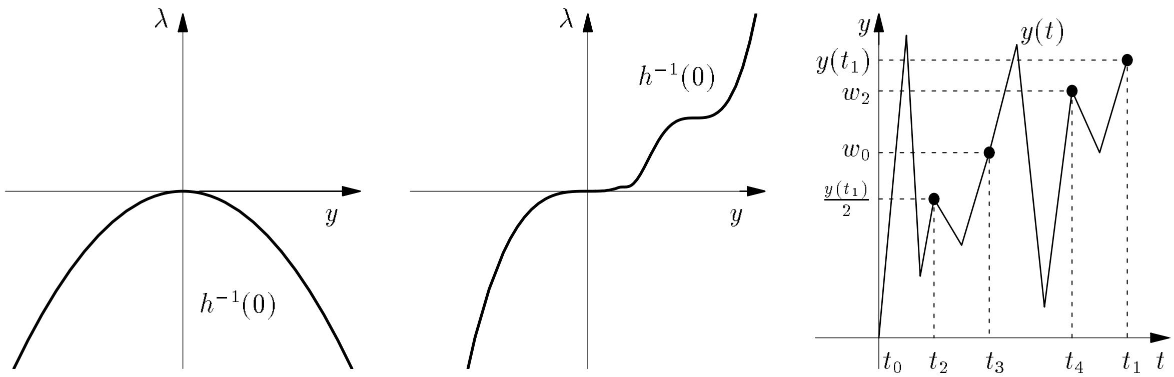
4.7. Example ([7])

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= 2x_1 \\ 0 &= x_2 - (x_1 + y^2)^2 \end{aligned} \tag{12}$$

Consider the point $((0,0),0)$. Obviously, $\varphi\colon]-1,1[\to \mathbb{R}^2 \times \mathbb{R} : t \mapsto ((t,t^2),0) \in S_{C^1,((0,0),0,0)}$, and hence, $((0,0),0) \in P_{C^1}$.
As can be easily seen,

$$h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (\lambda, y) \mapsto -(\lambda + y^2)^2$$

is a cut mapping of (12), and $(0,0)$ is a right limit point as well as a right PLP of $h^{-1}(0)$ (see Fig. 3(a)). Theorem 4.4. is not applicable, since $((0,0),0)$ does have the tangential property. Further, this example contradicts the "$\Longleftarrow$"-part of [2, Theorem 1]. $\square$



(a) (b) (c)

Figure 3: (a) $h^{-1}(0)$ with h from Example 4.7.. (b) $h^{-1}(0)$ with h from Example 4.8.. (c) Illustration of step (iv) of Example 4.8..

4.8. Example:

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= 2x_1 \\ 0 &= x_1 - \tilde{g}(x_2, y) \end{aligned} \tag{13}$$

where

$$\tilde{g}(x_2, y) = \int_0^y \alpha(x_2, w)\, dw$$

and

$$\alpha(x_2, w) = w^2 \left\{ \begin{array}{ll} 0 & \text{if } w = 0 \\ 1 & \text{if } w < 0 \\ 1 + (1 + x_2^2)\cos\left(\frac{1}{w}\right) & \text{if } w > 0 \end{array} \right\}.$$

We now observe the following:

(i) The right hand sides of (13) are continously differentiable:
α is clearly continous at $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$. Let now $(z_n, w_n)_{n \in \mathbb{N}}$ be some sequence in $\mathbb{R} \times \mathbb{R}$

converging to $(z_0, 0)$ for some $z_0 \in \mathbb{R}$. Then, $|\alpha(z_n, w_n) - \alpha(z_0, 0)| \leq w_n^2(2 + z_n^2)$, and hence, α is continous on $\mathbb{R} \times \mathbb{R}$. Further, $D_1\alpha$ exists,

$$D_1\alpha(x_2, w) = \left\{ \begin{array}{ll} 0 & \text{if } w \leq 0 \\ 2x_2 w^2 \cos\left(\frac{1}{w}\right) & \text{otherwise} \end{array} \right\},$$

and hence, $D_1\alpha$ is continous on $\mathbb{R} \times \mathbb{R}$. So, we get that $\tilde{g} \in C^1$ ([3, 8.11., Exercise 1]).

(ii) $(0, 0)$ is neither a right nor a left PLP of $h^{-1}(0)$, if h is a cut mapping of (13) at $((0, 0), 0)$: Clearly,

$$h\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (\lambda, y) \mapsto \lambda - \tilde{g}(0, y)$$

is a cut mapping of (13) at $((0, 0), 0)$ (see Fig. 3(b)). $\alpha(0, \cdot)$ is non-negative and is equal to 0 on a countable set only. Thus, $\tilde{g}(0, \cdot)$ is strongly monotone increasing. Hence, it is an injective, open mapping, and thus, $\tilde{g}(0, \cdot)^{-1}$ is continous and its domain contains some open ball with center 0.

(iii) $\varphi\colon \mathbb{R}_- \setminus \{0\} \to \mathbb{R}^2 \times \mathbb{R} : t \mapsto ((t, t^2), -\sqrt{-3t}) \in S_{C^1}$ and $\lim_{t \to 0} \varphi(t) = ((0, 0), 0)$.

(iv) There is no solution passing through $((0, 0), 0)$ (see Fig. 3(c) for illustration):
Assume $\psi = \langle\langle x_1, x_2\rangle, y\rangle \in S_{((0,0),0,t_0)}$ for some $t_0 \in \mathbb{R}$, contrary to the assertion. Then:

(a) $x_1^2(t) = x_2(t)$ holds for all $t \in \operatorname{dom}\psi$:
Set $\delta(t) := x_2(t) - x_1^2(t)$, then $\delta \in C^1$, $\delta(0) = 0$ and $\dot{\delta}(t) = 0$ for all $t \in \operatorname{dom}\psi$.

(b) Let, without loss of generality, $\operatorname{dom}\psi =]t_0 - \varepsilon, t_0 + \varepsilon[$ for some $t_0 \in \mathbb{R}$ and some $\varepsilon > 0$. Then we have

$$t - t_0 = \int_0^{y(t)} \alpha((t - t_0)^2, w)\, dw \tag{14}$$

for all $t \in \operatorname{dom}\psi$.

(c) $\exists_{t_1 \in]t_0, t_0 + \varepsilon[} \exists_{k \in \mathbb{Z}_+} \frac{1}{(2k+1)\pi} \in]\frac{y(t_1)}{2}, y(t_1)[$:
Let $t \in]t_0, t_0 + \varepsilon[$ and $y(t) < 0$. By definition of α, one gets $x_1(t) = t - t_0 = \frac{1}{3}y^3(t) < 0$. Contradiction. Let now $t \in]t_0, t_0 + \varepsilon[$ and $y(t) = 0$. Then, $x_1(t) = 0$. Contradiction. Hence, there is some $t_1 \in]t_0, t_0 + \varepsilon[$ that $y(t_1) \in]0, \frac{1}{3\pi}[$. Set

$$k := 1 + \sup\{q \in \mathbb{Z} |\ y(t_1) \leq \frac{1}{(2q + 1)\pi}\}.$$

Obviously, $k \in \mathbb{Z}$ and $k \geq 2$, as well as $\frac{1}{(2k+1)\pi} \leq y(t_1)$ and

$$\begin{aligned} y(t_1) &\leq \frac{1}{(2k - 1)\pi} \\ &\leq \frac{2}{(4k - 2)\pi} \\ &\leq \frac{2}{(2k + 1 + 2k - 3)\pi} \\ &\leq \frac{2}{(2k + 1)\pi}. \end{aligned}$$

(d) $\exists_{t_2 \in]t_0, t_1[} y(t_2) = \frac{y(t_1)}{2}$.

(e) Set $w_0 := \frac{1}{(2k+1)\pi}$. Then $\exists_{w_1 \in]w_0, y(t_1)[} \forall_{w \in [w_0, w_1]} \cos\left(\frac{1}{w}\right) < 0$.

(f) $\alpha((\cdot - t_0)^2, w)$ is monotone decreasing for all $w \in [w_0, w_1]$:

$$\alpha((t-t_0)^2, w) = w^2(1 + (1 + (t-t_0)^4)\underbrace{\cos\left(\frac{1}{w}\right)}_{<0})$$

Especially for $t \in [t_2, t_1]$ one gets

$$\alpha((t-t_0)^2, w) \leq \alpha((t_2-t_0)^2, w). \tag{15}$$

(g) $\exists_{w_2 \in]w_0, w_1[} \forall_{w \in [w_0, w_2]} \alpha((t_2 - t_0)^2, w) < 0$, since $\alpha \in C^0$.

(h) By (iv-f) we have $\forall_{w \in [w_0, w_2]} \forall_{t \in [t_2, t_1]} \alpha((t-t_0)^2, w) < 0$.

(i) $\exists_{t_3 \in]t_2, t_1[} y(t_3) = w_0$ and $\exists_{t_4 \in]t_3, t_1[} y(t_4) = w_2$.

(j) By (14) we get

$$\begin{aligned} t_4 - t_3 &= (t_4 - t_0) - (t_3 - t_0) \\ &= \int_0^{w_2} \alpha((t_4 - t_0)^2, w)\, dw - \int_0^{w_0} \alpha((t_3 - t_0)^2, w)\, dw \\ &= \int_0^{w_0} \underbrace{\alpha((t_4 - t_0)^2, w) - \alpha((t_3 - t_0)^2, w)}_{<0 \text{ by (f)}} dw + \int_{w_0}^{w_2} \underbrace{\alpha((t_4 - t_0)^2, w)}_{<0 \text{ by (h)}} dw \\ &< 0. \end{aligned}$$

Contradiction.

So, $((0,0),0)$ is a FIP-1 of (13), but $(0,0)$ is not a PLP of $h^{-1}(0)$, which contradicts the "$\Longrightarrow$"-part of [2, Theorem 1][13]. Note that $((0,0),0)$ does not have the tangential property, i.e., conditions (i), (ii), and (iii) in Theorem 4.4. are fulfilled. □

Appendices

A Ordered Sets and Zorn's Lemma

A.1. Definition (Order, Ordered set) *Let A be a set and $R \subseteq A \times A$ a relation. R is called* order on A *and (A, R) is called* ordered set $:\Longleftrightarrow$

$$\begin{aligned} \forall_{a \in A} aRa &\qquad (reflexivity) \\ aRb \wedge bRa \Longrightarrow a = b &\qquad (antisymmetry) \\ aRb \wedge bRc \Longrightarrow aRc &\qquad (transitivity) \end{aligned}$$

Let now (A, R) be an ordered set.
$m \in A$ is a maximal element of (A,R) $:\Longleftrightarrow (mRa \Longrightarrow a = m)$.
(A, R) is called chain $:\Longleftrightarrow \forall_{a,b \in A}(aRb \vee bRa)$.
Let $C \subseteq A$ and $(C, R \cap C \times C)$ be a chain. $u \in A$ is called upper bound of C in (A, R) $:\Longleftrightarrow \forall_{c \in C} cRu$.
□

[13] [7, Example 3.6] is a counterexample of the "$\Longrightarrow$"-part of [2, Theorem 1], which has more practical importance and is even C^∞.

Throughout the paper we accepted the *Axiom of Choice* [8] as one of our axioms. In that case, the following holds:

A.2. Lemma (Zorn's Lemma [8]) *Let (A, R) be an ordered set for which every chain $(C, R \cap C \times C)$ with $C \subseteq A$ has an upper bound in A, then there is a maximal element of (A, R).* $\square$

B An Implicit Function Theorem

B.1. Theorem: *E, Q, Z be banach spaces over $\mathbb{R}$, and U be an open neighbourhood of $(x_0, y_0) \in E \times Q$. Let further $F\colon U \to Z$ and the following conditions be satisfied:*

(i) $F(x_0, y_0) = 0$

(ii) *D_2F exists as a partial F-derivative on U and $D_2F(x_0, y_0)$ is bijective*

(iii) *F and D_2F are continous at (x_0, y_0).*

Then

(a) $\exists_{r,r_0>0}\forall_{x\in B(x_0,r_0)}\exists!_{y\in B(y_0,r)}F(x,y) = 0.$
In the following, the corresponding mapping $x \mapsto y$ is denoted by $y(\cdot)$.

($\tilde{a}$) *$y(\cdot)$ is continous at x_0.*

(c) *If F is continous on some neighbourhood of (x_0, y_0), then $y(\cdot)$ is continous on some neighbourhood of x_0.*

(e) *If F is F-differentiable at (x_0, y_0), then $y(\cdot)$ is F-differentiable at x_0 and*
$Dy(x_0) = -D_2F(x_0, y_0)^{-1} \circ D_1F(x_0, y_0).$ $\square$

Proof: (a) and (c) are exactly the same as the corresponding assertions of [8, Theorem 4.B]. We have to deal with ($\tilde{a}$) and (e):

($\tilde{a}$) Assume that there is some sequence $(x_n)_{n\in\mathbb{N}}$ in $B(x_0, r_0)$ with $x_n \to x_0$, but $y(x_n) \not\to y_0$, contrary to the assertion. Then, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and some $\varepsilon > 0$ that

$$\forall_{k\in\mathbb{N}} y(x_{n_k}) \notin B(y_0, \varepsilon). \tag{16}$$

Now, apply (a) to $F|_{B(x_0,r_0)\times B(y_0,\varepsilon)}$:

$$\exists_{\tilde{r},\tilde{r}_0>0}\forall_{x\in B(x_0,\tilde{r}_0)}\exists!_{\tilde{y}\in B(y_0,\tilde{r})}F(x,\tilde{y}) = 0. \tag{17}$$

In the following, the corresponding mapping $x \mapsto \tilde{y}$ is denoted by $\tilde{y}(\cdot)$.
We find some $q \in \mathbb{N}$ with $x_{n_q} \in B(x_0, \tilde{r}_0)$, since $\lim\limits_{k\to\infty} x_{n_k} = x_0$. By (17) we have $\tilde{y}(x_{n_q}) \in B(y_0, \tilde{r}) \subseteq B(y_0, \varepsilon) \subseteq B(y_0, r)$ and $F(x_{n_q}, \tilde{y}(x_{n_q})) = 0$. Thus, by (16), we have found two different zeros, $(x_{n_q}, y(x_{n_q}))$ and $(x_{n_q}, \tilde{y}(x_{n_q}))$, of F, with $x_{n_q} \in B(x_0, r_0)$ and $y(x_{n_q}), \tilde{y}(x_{n_q}) \in B(y_0, r)$. Contradiction.
(e) Let $\Delta x := x - x_0$ and $\Delta y := y(x) - y_0$. We know that $y(\cdot)$ is continous at x_0 by ($\tilde{a}$). Further, F is F-differentiable at (x_0, y_0). Thus, for any $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x \in B(x_0, \delta)$ the following is true:

$$\| \underbrace{F(x, y(x))}_{=0} - \underbrace{F(x_0, y_0)}_{=0} - DF(x_0, y_0)(\Delta x, \Delta y)\| \le \varepsilon\|(\Delta x, \Delta y)\|.$$

From [8, Prop. 4.14.a] it follows

$$\|D_1F(x_0,y_0)\Delta x + D_2F(x_0,y_0)\Delta y\| \le \varepsilon\|(\Delta x,\Delta y)\|.$$

$D_2F(x_0,y_0)^{-1}$ is continous by Banach's open mapping Theorem [8]. Thus

$$\|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\Delta x + \Delta y\| \;\le \varepsilon \underbrace{\|D_2F(x_0,y_0)^{-1}\|}_{M:=} \cdot \|(\Delta x,\Delta y)\|.$$

The aim is to get the right hand side of the foregoing inequality free of Δy:
We first choose $\|(\Delta x,\Delta y)\| := \|\Delta x\| + \|\Delta y\|$ [3, 3.20.] and obtain

$$\|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\Delta x + \Delta y\| \;\le \varepsilon M(\|\Delta x\| + \|\Delta y\|). \tag{18}$$

From that it follows

$$\begin{aligned}\|\Delta y\| &\le \|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\Delta x + \Delta y\| \; + \|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\Delta x\| \\ &\le \varepsilon M(\|\Delta x\| + \|\Delta y)\|) \; + \|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\| \cdot \|\Delta x\|.\end{aligned}$$

We assume $\varepsilon M \le \frac{1}{2}$ without loss of generality, and it follows

$$\|\Delta y\| \le \frac{1}{2}\|\Delta x\| + \frac{1}{2}\|\Delta y\| \; + \|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\| \cdot \|\Delta x\|$$

and hence

$$\|\Delta y\| \le \|\Delta x\| \cdot \left(1 + 2\,\|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\|\right).$$

From (18) we get

$$\|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\Delta x + \Delta y\| \le \varepsilon M\|\Delta x\| \cdot \left(2 + 2\,\|D_2F(x_0,y_0)^{-1}\circ D_1F(x_0,y_0)\|\right)$$

as long as $x \in B(x_0,\delta)$, and the proof is complete. □

C Proof of Lemma 4.5.

C.1. Lemma: *E, F be banach spaces over $\mathbb{R}$, $f\colon U \to F \in C^1$, $U \subseteq E$ open, V an open, convex, nonempty set, $\overline{V} \subseteq U$. Then*

$$\forall_{a,b,x_0\in V}\|f(b) - f(a) - Df(x_0)(b-a)\| \;\le \|b-a\| \sup_{x\in V}\|Df(x) - Df(x_0)\|$$

□

Proof: Since V is convex, we have $S := \mathrm{co}(\{a,b\}) \subseteq V$. By [3, 8.6.2.] and $S \subseteq V$ one gets the assertion[14]. □

Proof (of Lemma 4.5.) (i) trivial.

(ii) We set $v := D_1g(p)f(p) + D_3g(p)1 \in \mathbb{R}^k$, $x_t := x(t+t_0)$, $y_t := y(t+t_0)$, $\|(s,t,z)\| := |s| + |t| + \|z\|$ [3, 3.20.], and show that

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{(s,t,z)\in B(0,\delta)} \; \underbrace{\|G(s,t,z) - T(s,t,z)\|}_{H:=} \le \varepsilon\|(s,t,z)\| \tag{19}$$

[14] $\mathrm{co}(M)$ denotes the convex hull of M.

for $T(s,t,z) := (s-t)v + D_2g(p)z$:
Let $\varepsilon \in]0,1[$ and choose $\delta' > 0$ so that $\overline{B(p,\delta')} \subseteq U$ and

$$\sup_{q \in B(p,\delta')} \|Dg(q) - Dg(p)\| < \varepsilon. \tag{20}$$

This choice is practicable because $g \in C^1(U)$. Let $b := (x_0 + s \cdot f(p), y_t + z, s + t_0)$ and $a := (x_t, y_t, t + t_0)$. Then, there is some $\delta > 0$ so that $a, b \in B(p, \delta')$ and

$$\|x_t - x_0 - t \cdot f(p)\| \le \varepsilon |t| \tag{21}$$

as long as $(s,t,z) \in B(0,\delta)$ (because of continuity and differentiability, respectively). From (21) it follows

$$\begin{aligned} \|x_t - x_0 - s \cdot f(p)\| &\le \|x_t - x_0 - t \cdot f(p)\| \ + \|(s-t) f(p)\| \\ &\le \varepsilon |t| + |s-t| \cdot \|f(p)\|. \end{aligned} \tag{22}$$

Set

$$\begin{aligned} K :&= G(s,t,z) - Dg(p)(x_0 + s \cdot f(p) - x_t, z, s-t) \\ &= G(s,t,z) - s \cdot v + D_3 g(p) t \ + D_1 g(p)(x_t - x_0) - D_2 g(p) z. \end{aligned}$$

Then, by (20), (21), (22), and Lemma C.1. one gets

$$\begin{aligned} \|K\| &\le \varepsilon \|(x_0 + s \cdot f(p) - x_t, z, s-t)\| \\ &\le \varepsilon \left(\|(x_t - x_0 - s \cdot f(p)\| + \|z\| + |s-t| \right) \\ &\le \varepsilon^2 |t| + \varepsilon |s-t| \cdot \|f(p)\| + \varepsilon (\|z\| + |s-t|) \\ &\le \varepsilon M \cdot \|(s,t,z)\| \end{aligned}$$

for some $M > 0$ and all $(s,t,z) \in B(0,\delta)$. So, by definition of H, we have

$$\begin{aligned} H &= \|G(s,t,z) - s \cdot v - D_2 g(p) z \ + D_1 g(p) f(p) t + D_3 g(p) t\| \\ &= \|K - D_1 g(p)(x_t - x_0) + D_1 g(p) f(p) t\| \\ &\le \|K\| + \|D_1 g(p)\| \cdot \|x_t - x_0 - t \cdot f(p)\| \\ &\le \|K\| + \|D_1 g(p)\| \cdot \varepsilon |t| \\ H &\le \varepsilon \tilde{M} \cdot \|(s,t,z)\| \end{aligned}$$

as long as $(s,t,z) \in B(0,\delta)$.

(iii) Set $s = 0$ and $z = 0$ in (19).

(iv) $D_{1,3}G$ exists as a partial F-derivative on W by the chain rule [8, Prop. 4.10.(a)] and is continous.
Setting $t = 0$ in (19) completes the proof. □

D Limit Points

D.1. Definition (Limit point [2][7]) *Let* $h \colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \in C^0$, $(\lambda_0, y_0) \in h^{-1}(0)$. (λ_0, y_0) *is called*

right limit point of $h^{-1}(0)$ $:\Longleftrightarrow \exists_{N \in \mathcal{U}((\lambda_0, y_0))} N \cap h^{-1}(0) \cap ((\lambda_0 + \mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}^m) = \emptyset$

left limit point of $h^{-1}(0)$ $:\Longleftrightarrow \exists_{N \in \mathcal{U}((\lambda_0, y_0))} N \cap h^{-1}(0) \cap ((\lambda_0 - \mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}^m) = \emptyset$ □

As shown in [7, Lemma A.2], this definition is equivalent to that given in [2].

References

[1] H. Amann. *Gewöhnliche Differentialgleichungen*, volume 75 of *Applied Mathematical Sciences*. Walter de Gruyter, Berlin and New York, 1983. de-Gruyter-Lehrbuch.

[2] L. O. Chua and A.-C. Deng. Impasse points. Part I: Numerical aspects, Part II: Analytical aspects. *International Journal of Circuit Theory and Applications*, 17:213–235, 271–282, 1989.

[3] J. Dieudonné. *Foundations of Modern Analysis*, volume 1. Academic Press, New York and London, 7th edition, 1968.

[4] E. Griepentrog and R. März. *Differential-Algebraic Equations and Their Numerical Treatment*, volume 88 of *TEUBNER-TEXTE zur Mathematik*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, first edition, 1986.

[5] R. K. Miller and A. N. Michel. *Ordinary Differential Equations*. Academic Press, 1982.

[6] S. Reich. On a geometrical interpretation of differential-algebraic-equations. *Circuits, Systems, and Signal Processing*, 9(4):367–382, 1990.

[7] G. Reißig. Impasse points: Examples and counterexamples. TU Preprint ET-IEE-1-1993, TU Dresden, 19 July 1993. available from author.

[8] E. Zeidler. Part I: Fixed-point theorems. In *Nonlinear Functional Analysis and its Applications*. Springer-Verlag, New York, Berlin etc., corrected edition, 1993.

24 Jan. 1994